Upper and lower bounds for stochastic processes

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- (Borell) $\Rightarrow$ density $f(x) = e^{-U(x)}$, U-convex on an affine subspace.
- Isotropic position $\mathbf{E}X_i = 0$, $\mathbf{E}X_i^2 = 1$ $\mathbf{E}X_iX_j = 0$, $i \neq j$.

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- Uniform: $(X_1, X_2, ..., X_m)$, on K-convex body.

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- Why: $\mathbf{E}\|X\|$ for any norm $\| \cdot \|$ on $\mathbb{R}^m$, i.e.

$$\mathbf{E} \sup_{t \in T} X_t = \mathbf{E} \sup_{t \in \overline{\mathrm{conv}}\, T} X_t.$$

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- **Theorem (1987 Fernique-Talagrand)**
There exists universal K s.t.

$$K^{-1}\gamma_2(T, d_2) \leqslant \mathbf{E}\sup_{t\in T} X_t \leqslant K\gamma_2(T, d_2).$$

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- Integration by parts

$$g(T) \leqslant \mathbf{E} \sup_{n \geqslant 1} |G_{t^n}| = M \int_0^\infty \mathbf{P}(\sup_{n \geqslant 1} |G_{t_n}| \geqslant Mu) du \leqslant$$

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- Talagrand's Theorem $\Rightarrow \exists t^n \in \mathbb{R}^m$, $n \geqslant 1$ s.t. $T - T \subset \overline{\mathrm{conv}}\{t^n : \ n \geqslant 1\}$ and $\|G_{t^n}\|_{\log(n+2)} \leqslant Kg(T)$.

Tools

- Sudakov minoration: $T = \{t^1, ..., t^N\}$ $d_2(t^i, t^j) \geqslant a$, $i \neq j \Rightarrow$

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- Gaussian concentration: let $\sigma = \sup_{t \in T} d_2(t, 0)$

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$$\mathbf{E}\sup_{t \in T} G_t \leqslant (\mathbf{E}\sum_{t \in T} |G_t|^p)^{\frac{1}{p}} \leqslant KA$$

and $A = \sup_{t \in T} 2^{n/2} d_2(t, 0)$ if $p = 2^n$.

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Remark

If $p = 2^n$, $\|t^i\|_2 \leqslant C_1 \frac{A}{\sqrt{p}}$ and $\|t^i - t^j\|_2 \geqslant C_2 \frac{A}{\sqrt{p}}$ for $i \neq j$ then

$$K^{-1} \sup_{t \in T} 2^{n/2} d_2(t, 0) \leqslant g(T) \sim A \leqslant K \sup_{t \in T} 2^{n/2} d_2(t, 0).$$

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- $g(T) \sim \inf_\sigma \sup_{n \geqslant 1} a_{\sigma(n)} \sqrt{\log(n+2)}$.

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- $T = \Sigma_n \cap B_2(0, 1)$, where $B_2^m(0, 1) = \{x \in \mathbb{R}^m : \|x\|_2 \leqslant 1\}$.

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Examples: $\gamma_2(T, d_2)$

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Gaussian comparison

- Suppose $(X_t)_{t \in T}$, $T \subset \mathbb{R}^m$ satisfies

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- $T \subset \mathbb{R}^m$, $(\varepsilon_i)_{i=1}^m$ Bernoulli sequence

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- Point-wise estimate

$$b(T) = \mathbf{E} \sup_{t \in T} \sum_{i=1}^{m} t_i \varepsilon_i \leqslant \sup_{t \in T} \sum_{i=1}^{m} |t_i| = \|t\|_1.$$

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- **Theorem (2013 W.B. and R. Latala)**
  There exists a universal constant K and sets $T_1, T_2 \subset \ell^2$ such that $T \subset T_1 + T_2$ and

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Tools

- Sudakov-type minoration: $T = \{t^1, t^2, ..., t^N\}$, $d_2(t_i, t_j) \geqslant a$ and $\|t^i\|_\infty \leqslant b \Rightarrow$

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- **Remark**
If $\|X_{t^i}\|_p \leqslant C_1 A$, $\|X_{t^i} - X_{t^j}\|_p \geqslant C_2^{-1}A\, \forall i \neq j$ then $b(T) \sim A$.

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- How to determine $I(t)$?

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Theorem (2013 W.B. and R. Latala)

Let $(X_i)_{i=1}^m$ be a sequence of independent symmetric variables valued in $(\mathbb{R}^n, \|\cdot\|)$, let $\mathcal{C}$ be a countable VC-class of subsets of $\{1, 2, ..., m\}$ of dimension d. Then

$$\mathbf{E} \sup_{C \in \mathcal{C}} \|\sum_{i \in C} X_i\| \leqslant K\sqrt{d}\mathbf{E}\|\sum_{i=1}^{m} X_i\|.$$

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Kwapien question

- $(\mathbb{R}^m, \|\cdot\|)$, $x^i \in \mathbb{R}^m$, $i \geqslant 1$.
- Whether $\exists y^i, z^i \in \mathbb{R}^m$, for $i \geqslant 1$ and K-universal s.t. $x^i = y^i + z^i$ and

$$\sup_{\varepsilon_i \in \pm 1} \|\sum_{i\geqslant 1} y^i \varepsilon_i\| \leqslant K\mathbf{E}\|\sum_{i\geqslant 1} x^i \varepsilon_i\|$$

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- $(\mathbb{R}^m, \|\cdot\|_\infty)$-Bernoulli Theorem.

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- For any finite $T \subset \mathbb{R}^m$

$$\mathbf{E} \sup_{t \in T} \sum_{i=1}^{m} \theta_i(t_i)\varepsilon_i \leqslant b(T) = \mathbf{E} \sup_{t \in T} \sum_{i=1}^{m} t_i \varepsilon_i.$$

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- Bernoulli Comparison: OK if $|x^*(x^i)| \leqslant C|x^*(y^i)|$ $\forall\ x^* \in \mathbb{R}^m$, $i \in \{1, ..., m\}$.
- Bernoulli Theorem: OK if $\{(x^*(y^i))_{i=1}^m :\ x^* \in \mathbb{R}^N\}$ spans $\mathbb{R}^m$. Impossible if $m > N$.

Exponential densities

- Let $(X_t)_{t\in T}$

$$X_t = \sum_{i=1}^{m} t_i \eta_i, \quad t \in T$$

and $(\eta_i)_{i=1}^{m}$ i.i.d. symmetric $\eta_i \sim \eta$ of density $a_\alpha e^{-x^\alpha}$, $\alpha \geqslant 1$.

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- Let $d_\beta(x,y) = \|x-y\|_\beta$ and
$$\gamma_\alpha(T, d_\beta) = \inf \sup_{t\in T} \sum_{n=0}^{\infty} 2^{n/\alpha} d_\beta(t, T_n),$$
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- If $1 \leqslant \alpha \leqslant 2$ ($\alpha^{-1} + \beta^{-1} = 1$) then
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- Also $\exists y_n \in \mathbb{R}^m$ and $C < \infty$ s.t. $T - T \subset \overline{\mathrm{conv}}(y^n : \ n \geqslant 1)$ and

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- **Theorem (1997 R. Latala)**

  Let N_i be of moderate growth then for any $T \subset \mathbb{R}^m$ then $\mathbf{E}\sup_{t\in T} X_t$ is comparable with

  $$\inf\{M : \forall n \geqslant 1 \exists t^n \in \mathbb{R}^m \ \ s.t.\ \ \|t^n\|_{\mathcal{N},\log(n+2)} \leqslant M, T - T \subset \overline{\mathrm{conv}}(t)\}.$$

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- **Corollary**
  If $T = \{t^1, ..., t^N\}$*,* $N \geqslant e^{p^2}$ *and*

$$\|X_{t^i} - X_{t^j}\|_p \geqslant A,$$

then $\mathbf{E} \sup_{t \in T} X_t \geqslant K^{-1} A$*.*

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- **Remark (2014 W.B.)**
  If for each $t \in T$ $\exists$ $k \in \mathbb{R}^m$ s.t.

$$\sum_{i=1}^{m}(t_i - s_i)k_i \geqslant C^{-1}A \ \text{ and } \ \mathbf{P}(\bigcap_{i=1}^{m} |X_i| \geqslant k_i) \geqslant e^{-p}$$

for all $s \in T$, $s \neq t$ then Sudakov minoration works if $N > e^{p \log p}$.

Concentration

- If B s.t. $\mathbf{P}(B) \geqslant \frac{1}{2}$ then

$$\mathbf{P}(X \in B + \alpha u B_2^m) \geqslant 1 - e^{-u}, \quad u > 0.$$

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- Does the concentration suffices for the minoration?

Thank you for your attention