On multivariate solutions of a class of stochastic equations.

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Cargèse, 27.05.2014

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Examples:

- If N is a fixed constant, $A_i = N^{-\frac{1}{2}}$, $B = 0$, then X has the normal distribution;
- If N is a fixed constant, $A_i = N^{-\frac{1}{\alpha}}$ $(\alpha < 2)$, $B = 0$, then X has the α - stable distribution;

$$X =_d \sum_{j=1}^{N} A_j X_j + B.$$

Motivation

- stability of interacting particle systems
- analysis of the Quicksort algorithm
- branching random walks
- If $N = 1$, X is a solution to random difference equation

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Motivation

- stability of interacting particle systems
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History:

- if $N = 1$, Kesten, Grincevicius, 70's
- Kahane, Peyrière, 1976
- Durrett, Liggett, 1983
- Guivarc'h, Liu, 90's
- ...

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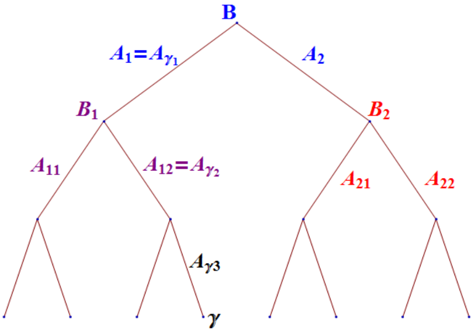
If $\mathbb{E} \log A < 0$ the series converges a.s. and its limit is a unique solution to $X =_d AX + B$. Indeed

$$\sum_{k=1}^{\infty} \prod_{j=1}^{k-1} A_j \cdot B_k = A_1 \Big(\sum_{k=2}^{\infty} \prod_{j=2}^{k-1} A_j \cdot B_k \Big) + B_1$$

In general case, we define

$$X = \sum_{n} \sum_{|\gamma|=n} \Pi_\gamma B_\gamma$$



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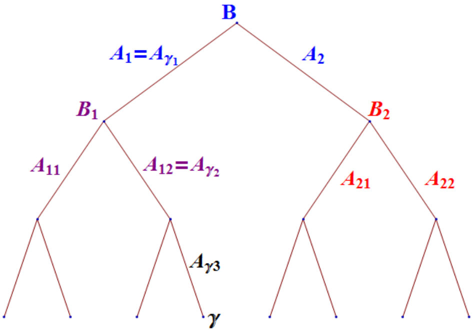
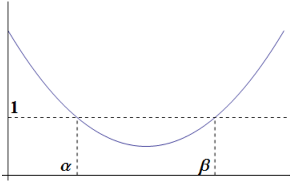


The series is convergent if $\mathbb{E}[\sum_{j=1}^N A^\alpha] < 1$ for some $\alpha < 1$. Then $(N = 2)$

$$X =_d A_1 X_1 + A_2 X_2 + B.$$

Description of solutions.

Let $\Phi(s) = \mathbb{E}\left[\sum_{j=1}^{N} A_1^s \right]$



Then, usually:

- $\mathbb{P}[X > t] \sim t^{-\alpha}$
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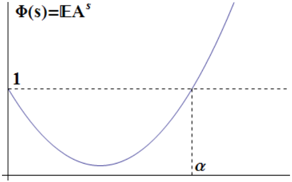
- if $\alpha = \beta < 1$, then $\mathbb{P}[X > t] \sim \log t \cdot t^{-\alpha}$

'Proof' of asymptotic for $N = 1$

$$X =_d AX + B$$

Theorem (Kesten 73, Goldie 91) Let X be a solution to $X =_d AX + B$. If $\mathbb{E}A^\alpha = 1$ and ..., then

$$\mathbb{P}[X > t] \sim C_+ t^{-\alpha}$$



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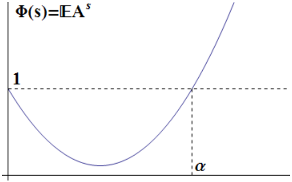

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Proof

$$\begin{aligned} f(s) &:= \mathbb{P}[X > e^s] = \mathbb{P}[AX + B > e^s] \\ &= \mathbb{P}[AX > e^s] + (\mathbb{P}[AX + B > e^s] - \mathbb{P}[AX > e^s]) \\ &= \mathbb{P}[X > e^{s - \log A}] + \phi(s). \end{aligned}$$

Let μ_A - the law of $\log A$. Then

$$f(s) = \mu_A * f(s) + \psi(s)$$

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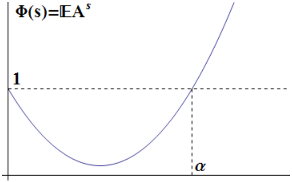

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Let $\mu_\alpha(dt) = e^{\alpha t}\mu_A(dt)$, $f_\alpha(s) = e^{\alpha s} f(s)$. Then

$$f_\alpha(s) = \mu_\alpha * f_\alpha(s) + \psi_\alpha(s)$$

and by the renewal theorem $f_\alpha(s) \to \frac{1}{m}\int \psi_\alpha(y)dy$

General case: Let X be a solution $X =_d \sum_{j=1}^{N} A_j X_j$.
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Define new random variables ($m = \mathbb{E}X$):

$$\begin{aligned} \mathbb{E}\big[f(\widetilde{X})\big] &= \frac{1}{m}\mathbb{E}\big[Xf(X)\big], \\ \mathbb{E}\big[f(\widetilde{A})\big] &= N\mathbb{E}\big[A_1 f(A_1)\big], \\ \mathbb{E}\big[f(\widetilde{B})\big] &= \mathbb{E}\Bigg[f\Bigg(\sum_{i=2}^{N} A_i X_i\Bigg)\Bigg]. \end{aligned}$$

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&= \frac{N}{m}\mathbb{E}\Big[A_1 X_1 f\Big(A_1 X_1 + \sum_{i=2}^{N} A_i X_i\Big)\Big] = \frac{N}{m}\mathbb{E}\big[A_1 X f(A_1 X + \widetilde{B})\big] \\
&= \mathbb{E}\big[f(\widetilde{A}\widetilde{X} + \widetilde{B})\big].
\end{aligned}$$

and, under some assumptions

$$\mathbb{P}[X > t] \sim t^{-\beta}$$

Multidimensional case:

$$X =_d \sum_{j=1}^{N} A_j X_j + B.$$

A_j are matrices, X_j, B vectors and we assume that A_j are i.i.d and N is constant.

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Asymptotic behavior can be proved only for very particular classes of matrices:

1. A_i have positive coefficients;
2. the law of A_i is absolutely continuous and A_i act irreducible on the sphere: $\forall x \in S$ $\forall$ open $U \subset S$: $\mathbb{P}[A_n \dots A_1 \cdot x \in U] > 0$;
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$N = 1$: Kesten, Guivarc'h, Le Page
$N > 1$: Damek, Guivarc'h, Mentemeier, Mirek, B.

$$X =_d \sum_{i=1}^{N} A_i X_i$$

Asymptotic. Define $\kappa(s) = \lim_{n\to\infty} \left(\mathbb{E}\|A_1..A_n\|^s\right)^{\frac{1}{n}}$

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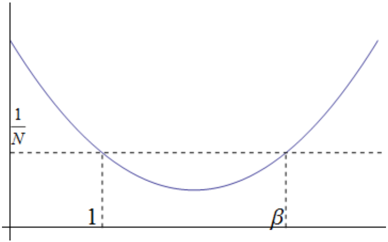
Theorem. If

- A_i are matrices with positive coefficients
- $\frac{1}{N}$ is the dominant eigenvalue of $\mathbb{E}A$ (then $\kappa(1) = \frac{1}{N}$)
- $\kappa(\beta) = \frac{1}{N}$ for $\beta > 1$



Then for every $u \in S^+ = S^{d-1} \cap (\mathbb{R}^+)^d$

$$\lim_{t\to\infty} t^\beta \mathbb{P}[\langle X, u\rangle > t] = e^\beta(u).$$

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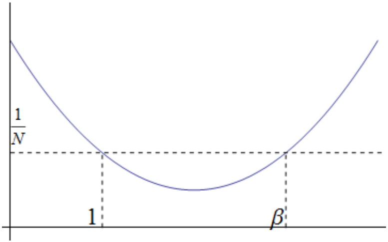

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Problems in the multidimensional case:

- change of the measure
- renewal theorem

Sketch of the proof

We denote by $A \cdot x$ the action of A on S^+: $A \cdot x = \frac{Ax}{|Ax|}$. For any function ϕ on S^+ we define transfer operators:

$$P^s\phi(x) = \mathbb{E}\big[|Ax|^s\phi(A \cdot x)\big] = \int |ax|^s\phi(a \cdot x)\mu(da)$$

Then there exists a unique function $e^s(x)$ such that $P^s e^s(x) = \kappa(s)e^s(x)$.

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Then there exists a unique function $e^s(x)$ such that $P^s e^s(x) = \kappa(s)e^s(x)$. We define Markovian operators on S^+:

$$\begin{aligned} Q^s\phi(x) &= \frac{1}{\kappa(s)e^s(x)}P^s(e^s\phi)(x) = \mathbb{E}\left[\phi(A\cdot x)|Ax|^s\frac{e^s(A\cdot x)}{\kappa(s)e^s(x)}\right] \\ &= \mathbb{E}\big[\phi(A\cdot x)q^s(x,A)\big] \\ (Q^s)^n\phi(x) &= \mathbb{E}\left[\phi(A_n..A_1\cdot x)|A_n..A_1x|^s\frac{e^s(A_n..A_1\cdot x)}{\kappa^n(s)e^s(x)}\right] \end{aligned}$$

One proves that $(Q^s)^n$ converges to the stationary measure π_s.

Markov chain on $S^+ \times \mathbb{R}$.

$$Q^\beta \phi(x) = \mathbb{E}\left[\phi(A \cdot x)|Ax|^\beta \frac{e^\beta(A \cdot x)}{\kappa(\beta)e^\beta(x)}\right] = \mathbb{E}\big[\phi(A \cdot x)q^\beta(x, A)\big]$$

Fix $x \in S^+$. We consider the Markov chain on $S^+ \times G$:

- $(X_0, S_0) = (x, I)$,
- given (X_n, S_n) we pick up g_{n+1} with respect to the measure $q^\beta(X_n, g)\mu(dg)$ and define

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Put $V_n = \log|S_n x| = \sum_{i=1}^n \log|g_i X_{i-1}|$ then

$$(X_n, V_n)$$

is a Markov chain on $S^+ \times \mathbb{R}$ and $\mathbb{E}_x^\beta V_n > 0$.

$$X =_d \sum_{i=1}^{N} A_i X_i$$

Fix $u \in S^+$. ν_u - the law of $\langle X, u\rangle$, a probability measure on $\mathbb{R}^+$

$\eta_u(dt) = t\nu_u(dt)$. Define: $f(u,t) = \eta_u(t,\infty)$ and

$$f_\alpha(u,t) = t^{\beta-1} f(u,t).$$

We prove

$$f_\alpha(x,t) = \sum_{n=0}^{\infty} \mathbb{E}_x^\beta \psi(X_n, t - V_n)$$

for a 'very good' function ψ.

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Finally, by Kesten's renewal theorem

$$G(x,t) = \sum_{n=0}^{\infty} \mathbb{E}_x^\beta \psi(X_n, t - V_n) \to \frac{1}{\beta} \int_{S^+} \int_{\mathbb{R}} \psi(y,s) ds \pi^\beta(dy)$$