Transport proofs of variance estimates for some classes of log-concave vectors

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Outline

Based on joint works with C. Roberto and P.M. Samson and D. Cordero-Erausquin

I. Poincaré inequality - A brief presentation.

II. A characterization of Poincaré inequality in terms of dimension free concentration of measure.

III. Variance estimates via transport arguments.

I. Poincaré inequality - A brief presentation

Poincaré inequality

Definition. Poincaré inequality

A probability measure $\mu \in \mathcal{P}(\mathbf{R}^d)$ satisfies the Poincaré inequality with a constant $C > 0$ (**PI**(C)) if

$$\mathrm{Var}_\mu(f) \leq C \int \|\nabla f\|^2 \, d\mu,$$

for all (bounded) $f : \mathbf{R}^d \to \mathbf{R}$ of class $\mathcal{C}^1$.

Basic example. The standard Gaussian measure

$$\gamma_d(dx) = \frac{1}{(2\pi)^{d/2}} e^{-\|x\|^2/2} \, dx$$

on $\mathbf{R}^d$ satisfies **PI**(1).

A classical sufficient condition

Theorem

If $\mu(dx) = e^{-V(x)}\,dx$ is a probability measure on $\mathbf{R}^d$ with a smooth density, and if for some $\varepsilon \in (0,1)$, it holds

$$\liminf_{\|x\| \to \infty} \varepsilon \|\nabla V\|^2(x) - \Delta V(x) > 0$$

then μ verifies $\mathbf{PI}(C)$, for some $C > 0$.

Drawback : This result does not provide a good estimate of the constant C.

The KLS conjecture

A probability measure μ is said *log-concave* if it has a density of the form e^{-V} for some *convex* function $V : \mathbf{R}^d \to \mathbf{R} \cup \{+\infty\}$.

It is said *isotropic* if $\int x\, \mu(dx) = 0$ and for all $i \neq j$

$$\int x_i x_j \, \mu(dx) = 0 \qquad \text{and} \qquad \int x_i^2 \, \mu(dx) = 1.$$

'KLS' conjecture [Kannan-Lovasz-Simonovits '95]

There is an absolute constant C such that for any positive integer d, any isotropic log-concave probability measure μ on $\mathbf{R}^d$ satisfies $\mathbf{PI}(C)$.

Remark.

Best dimensional bounds $C = cd^{5/6}$ (Guédon-Milman '11) improved into $C = cd^{2/3} \log d$ (Eldan '12) (improving preceding results by Bobkov-Nazarov, Paouris, Klartag, Fleury-Guédon-Paouris,...)

Application 1

Consider a probability measure

$$\mu(dx) = e^{-V(x)}\, dx$$

on $\mathbf{R}^d$ with a smooth potential V and define the diffusion operator

$$Lf = \Delta f - \nabla f \cdot \nabla V, \qquad \forall f \in \mathcal{C}^2(\mathbf{R}^d).$$

Consider the semigroup $(P_t)_{t\geq 0}$ generated by L :

$$\frac{d}{dt} P_t(f) = LP_t(f), \qquad \forall t > 0.$$

Theorem

With the notation above, the following propositions are equivalent :

1. μ satisfies $\mathbf{PI}(C)$
2. For all f, $\mathrm{Var}_\mu(P_t f) \leq e^{-2t/C}\, \mathrm{Var}_\mu(f)$, for all $t \geq 0$.

Application 2 : Concentration of measure

Let μ be a probability measure on $\mathbf{R}^d$ and $\alpha : \mathbf{R}^+ \to [0,1/2]$ a non-increasing function.

Definition.

- One says that μ satisfies the concentration of measure property with the concentration profile α if for all $A \subset \mathbf{R}^d$ with $\mu(A) \geq 1/2$ it holds
$$\mu(A_t) \geq 1 - \alpha(t), \qquad t \geq 0,$$
where
$$A_t = \{x \in \mathbf{R}^d ; \inf_{a \in A} \|x - a\| \leq t\}, \qquad t \geq 0.$$

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  where

$$A_t = \{x \in \mathbf{R}^d ; \inf_{a \in A} \|x - a\| \leq t\}, \qquad t \geq 0.$$

- One says that μ satisfies the dimension-free concentration of measure property with the concentration profile α, if for all $n \in \mathbb{N}^*$, the product measure $\mu^{\otimes n}$ satisfies the concentration property with the profile α on $(\mathbf{R}^d)^n$ equipped with the Euclidean distance.

Application 2

Theorem [Gromov-Milman / Aida-Stroock / Bobkov-Ledoux]

Let $\mu \in \mathcal{P}(\mathbf{R}^d)$. If μ satisfies $\mathbf{PI}(C)$, then μ satisfies the dimension free concentration property with the exponential profile

$$\alpha(t) = b \exp\left(-\frac{a}{\sqrt{C}}t\right)$$

where a, b are universal constants.

II. A characterization of Poincaré inequality in terms of dimension free concentration

Main result

Goal : Identify the class of probability measures enjoying the dimension free concentration property.

Theorem [G.-Roberto-Samson '13]

Suppose that μ satisfies the dimension free concentration property with a concentration profile α such that $\alpha(t) < 1/2$ for at least one value of $t > 0$, then μ satisfies $\mathbf{PI}(C)$ with the constant

$$C = \left(\inf \left\{ \frac{t}{\overline{\Phi}^{-1}(\alpha(t))} : t \text{ s.t. } \alpha(t) < 1/2 \right\} \right)^2,$$

where

$$\overline{\Phi}(s) = \frac{1}{\sqrt{2\pi}} \int_s^{+\infty} e^{-u^2/2}\, du.$$

Completes previous results by Bobkov-Götze or Bobkov-Houdré identifying the class of 1-dimensional probability measures enjoying convex dimension free concentration or non-trivial dimension free concentration with respect to the ℓ_∞ metric.

Proof of the theorem

Three ingredients :

- An equivalent form of concentration of measure in terms of deviation inequalities for the functions $Q_s f$ defined by

$$Q_s f(x) = \inf_{y \in \mathbf{R}^d} \left\{ f(y) + \frac{1}{s} \|x - y\|^2 \right\}, \qquad x \in \mathbf{R}^d.$$

- The Hamilton-Jacobi equation $\frac{\partial}{\partial s} Q_s f(x) = -\frac{1}{4} \|\nabla Q_s f\|^2(x)$.
- Central limit theorem.

Concentration and deviations of Lipschitz functions

Lemma

The probability μ satisfies the concentration property with the profile α if and only if for all 1-Lipschitz function f the following deviation inequality holds

$$\mu(f > m_\mu(f) + r) \leq \alpha(r), \qquad \forall r \geq 0,$$

where $m_\mu(f)$ is a median of f under μ.

Concentration and inf-convolution.

For all $s > 0$, and all $f : \mathbf{R}^d \to \mathbf{R}$ bounded from below, define

$$Q_s f(x) = \inf_{y \in \mathbf{R}^d} \left\{ f(y) + \frac{1}{s}\|x-y\|^2 \right\}, \qquad x \in \mathbf{R}^d.$$

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Lemma

The probability μ satisfies the concentration property with the profile α if and only if for all function $f : \mathbf{R}^d \to \mathbf{R}$ bounded from below the following deviation inequality holds

$$\mu(Q_s f > m_\mu(f) + t) \leq \alpha(\sqrt{st}), \qquad t \geq 0, s > 0,$$

where $m_\mu(f)$ is a median of f under μ.

Concentration and inf-convolution.

For all $s > 0$, and all $f : (\mathbf{R}^d)^n \to \mathbf{R}$ bounded from below, define

$$Q_s f(x) = \inf_{y \in (\mathbf{R}^d)^n} \left\{ f(y) + \frac{1}{s} \|x - y\|^2 \right\}, \qquad x \in (\mathbf{R}^d)^n.$$

Lemma

The probability μ satisfies the dimension free concentration property with the profile α if and only if for all function $f : (\mathbf{R}^d)^n \to \mathbf{R}$ bounded from below the following deviation inequality holds

$$\mu^n(Q_s f > m_{\mu^n}(f) + t) \leq \alpha(\sqrt{st}), \qquad t \geq 0, s > 0,$$

where $m_{\mu^n}(f)$ is a median of f under μ.

Sketch of proof of the theorem

By assumptions, it holds

$$\mu^n \left(Q_s f > m_{\mu^n}(f) + t\right) \leq \alpha(\sqrt{st}), \qquad \forall n, \forall f, \forall s, t > 0.$$

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Apply this inequality with

$$f_n(x) = h(x_1) + h(x_2) + \cdots + h(x_n),$$

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Then choose

$$t = u^2\sqrt{n} \qquad \text{and} \qquad s = 1/\sqrt{n}$$

Sketch of proof of the theorem

$$\mathbb{P}\left(\frac{\sum_{i=1}^n Q_{1/\sqrt{n}}h(X_i)}{\sqrt{n}} > m_n + u^2\right) \leq \alpha(u), \qquad \forall n, \forall u > 0,$$

where X_i are i.i.d of law μ and m_n is the median of the random variable

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$$\mathbb{P}\left(\frac{\sum_{i=1}^{n} Q_{1/\sqrt{n}}h(X_i) - \mathbb{E}\left[Q_{1/\sqrt{n}}h(X_i)\right]}{\sqrt{n}} > m_n - \sqrt{n}\mathbb{E}\left[Q_{1/\sqrt{n}}h(X_1)\right] + u^2\right) \leq \alpha(u),$$

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Letting $n \to \infty$, we get

$$\mathbb{P}\left(Y > \frac{1}{4}\int \|\nabla h\|^2\, d\mu + u^2\right) \leq \alpha(u), \qquad \forall u \geq 0,$$

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Equivalently, with $\overline{\Phi}(t) = \frac{1}{\sqrt{2\pi}} \int_t^{+\infty} e^{-u^2/2} \, du$, it holds

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Replacing h by λh, $\lambda > 0$, and optimizing over λ yields

$$\overline{\Phi}^{-1}(\alpha(u))\sqrt{\mathrm{Var}_\mu(h)} \leq \sqrt{u^2 \int \|\nabla h\|^2 \, d\mu}$$

Formal comparison with a result by E. Milman

Theorem (E. Milman (09-10))

Suppose that $\mu(dx) = e^{-V(x)}\,dx$ is a log-concave probability measure on $\mathbf{R}^d$.

If μ satisfies the following concentration property : for some $t_o > 0$ and $\lambda_o \in [0, 1/2)$

$$\mu(A_{t_o}) \geq 1 - \lambda_o, \qquad \forall A \subset \mathbf{R}^d,\ \text{s.t. } \mu(A) \geq 1/2,$$

then μ satisfies $\mathbf{PI}(C)$ with $C = 4\left(\frac{t_o}{1-2\lambda_o}\right)^2$.

Remarks :

- This result extends on manifolds
- “Log-concave” $\leftrightarrow$ “dimension free concentration”.

Application : Reduction of the KLS conjecture

Theorem [E. Milman]

The K.L.S conjecture is equivalent to the following statement :

There exist $t_o > 0$ and $\lambda_o \in [0, 1/2)$ such that for any positive integer d, any d-dimensional isotropic and log-concave probability measure μ satisfies

$$\mu(A_{t_o}) \geq 1 - \lambda_o, \qquad \forall A \subset \mathbf{R}^d \text{ s.t. } \mu(A) \geq 1/2.$$

Remark.

Alternative proof using the characterization of Poincaré inequality in terms of dimension free concentration. Indeed, the class of isotropic log-concave probability measures is stable under tensor products.

III. Variance estimates via transport arguments

The KLS and Variance conjectures

Recall the KLS conjecture.

KLS conjecture

There exists a universal constant $c > 0$ such that any isotropic log-concave random vector X satisfies

$$\mathrm{Var}(f(X)) \leq c\mathbb{E}[\|\nabla f(X)\|^2], \qquad \text{for all f smooth enough.}$$

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Variance conjecture

There exists a universal constant $c > 0$ such that any d-dimensional isotropic log-concave random vector X satisfies

$$\mathrm{Var}(\|X\|^2) \leq cd.$$

Recent results

Klartag '09 The variance conjecture is true for unconditional isotropic log-concave random vectors. Alternative proof by Barthe-Cordero '13 using direct $\mathbb{L}_2$ methods.

Recall that a random vector X is said unconditional if

$$\mathrm{Law}(X) = \mathrm{Law}((\varepsilon_1 X_1, \ldots, \varepsilon_d X_d)), \qquad \forall \varepsilon_i = \pm 1.$$

Guédon-Milman '11 For any isotropic log-concave r.v. X,

$$\mathrm{Var}(\|X\|^2) \leq cd^{4/3}.$$

Eldan '13 If the variance conjecture is true then KLS is true up to a $\log d$ factor.

Main result

Goal : Recover (and slightly extend) Klartag's result about unconditional random vectors using optimal transport tools.

Notation : If X is a random vector, the random vector $\overline{X}$ is defined by

$$\overline{X}_i = X_i - \mathbb{E}[X_i|X_1, \ldots, X_{i-1}], \qquad \forall i \in \{1, \ldots, d\}.$$

One denotes by $\mathcal{F}_i$ the σ field $\sigma(X_1, \ldots, X_i)$ and $\mathcal{F}_0 = \{\emptyset, \mathbf{R}^d\}$.

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Theorem [G. - Cordero '14]

If X is a log-concave random vector, then the random vector $\overline{X}$ satisfies the following Poincaré type inequality

$$\mathrm{Var}(f(\overline{X})) \leq c \sum_{i=1}^{d} \mathbb{E}\left[\mathbb{E}[\overline{X}_i^2 | \mathcal{F}_{i-1}] \, (\partial_i f(\overline{X}))^2\right], \qquad \forall f \text{ sufficiently smooth}$$

where c is a universal constant.

Proof uses transport arguments.

Comments

$$(*) \qquad \mathrm{Var}(f(\overline{X})) \leq c \sum_{i=1}^{d} \mathbb{E}\left[\mathbb{E}[\overline{X}_i^2 | \mathcal{F}_{i-1}]\,(\partial_i f(\overline{X}))^2\right], \qquad \forall f \text{ sufficiently smooth}$$

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- The random vector $\overline{X}$ is the projection in the space $\mathbb{L}_2(\Omega, \mathbb{P}; \mathbf{R}^d)$ of X onto the subspace of *martingale increments* with respect to the filtration $(\mathcal{F}_i)_{0\leq i\leq d}$.

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- The operation $X \to \overline{X}$ does not preserve log-concavity.

Comments

Corollary

For any log-concave random vector X,

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If in addition X is isotropic, then $\mathrm{Var}(\|\overline{X}\|^2) \leq cd$ for some universal $c > 0$.

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  This construction can be iterated ...

Idea of the proof

Theorem [Bobkov-Gentil-Ledoux '01]

A probability measure μ on $\mathbf{R}^d$ satisfies $\mathbf{PI}(C)$ for some $C > 0$ if and only if there is some D such that it satisfies the transport-entropy inequality

$$\mathcal{T}(\nu,\mu) \leq \mathrm{H}(\nu|\mu), \qquad \forall \nu \in \mathcal{P}(\mathbf{R}^d),$$

where

$$\mathrm{H}(\nu|\mu) = \int \log \frac{d\nu}{d\mu}\, d\nu$$

$$\mathcal{T}(\nu,\mu) = \inf \mathbb{E}[\min(D\|X-Y\|; D^2\|X-Y\|^2)]$$

The link between C and D is quantitative.

General idea :
Poincaré type inequalities can be represented as transport-entropy inequalities.

Some tools used in the proof of the main result

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First tool : If μ is log-concave and $T : \mathbf{R}^d \to \mathbf{R}^d$ is a $\mathcal{C}^1$ diffeomorphism sending μ onto ν then

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Second tool : T = Knothe map.

Third tool : Poincaré inequality for one dimensional log-concave probability measures. If Y is a one dimensional log-concave random variable, then the Poincaré constant of Y satisfies

$$C_Y \simeq \mathrm{Var}(Y).$$