

Taut Strings Accompanying Wiener Process

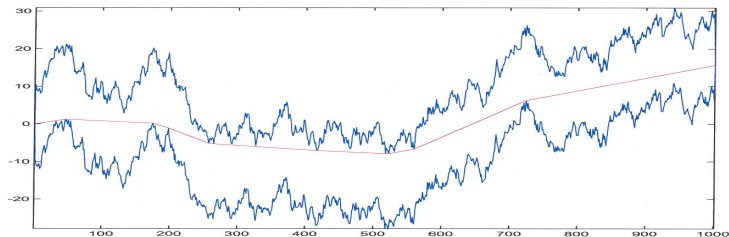
M.Lifshits (St.Petersburg and Linköping)

HDP Cargese, May, 2014

This is a joint work with [E. Setterqvist](#) from Linköping university, Sweden.

Taut string

$$\begin{cases} \int_0^T f'(t)^2 dt \searrow \min \\ f(0) = w(0), \quad f(T) = w(T), \\ w(t) - r \leq f(t) \leq w(t) + r, \quad 0 \leq t \leq T. \end{cases} \quad \text{or (!!)} \quad \int_0^T \varphi(f'(t)) dt \searrow \min$$



Formal setting

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Let W be a Wiener process. We are mostly interested in its approximation characteristics

$$I_W(T, r) := \inf\{|h|_T; h \in AC[0, T], \|h - W\|_T \leq r, h(0) = 0\}$$

and

$$I_W^0(T, r) := \inf\{|h|_T; h \in AC[0, T], \|h - W\|_T \leq r, h(0) = 0, h(T) = W(T)\}.$$

Main results

Theorem

There exists $C \in (0, \infty)$ such that for any $q > 0$ if $\frac{r}{\sqrt{T}} \rightarrow 0$, then

$$\frac{r}{T^{1/2}} I_W(T, r) \xrightarrow{L_q} C \quad \text{and} \quad \frac{r}{T^{1/2}} I_W^0(T, r) \xrightarrow{L_q} C.$$

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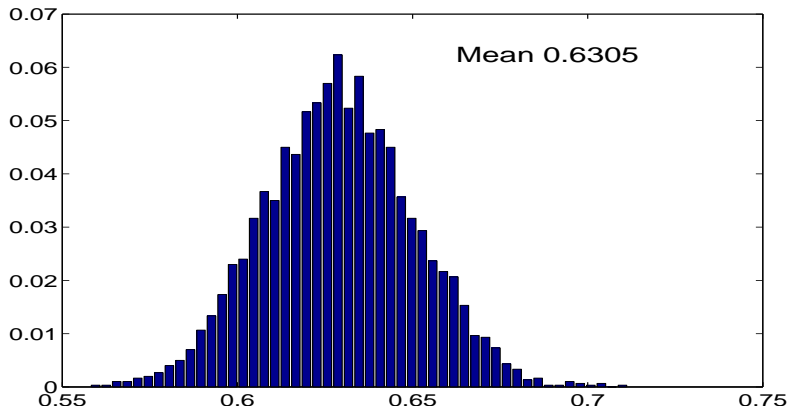
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Empirical modelling of \mathcal{C}

EJDEid.com



$$\mathcal{C} \approx 0.63$$

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$$\mathbb{P} (|I_W - \text{med}(I_W)| > \rho) \leq \mathbb{P} (|N| > \rho)$$

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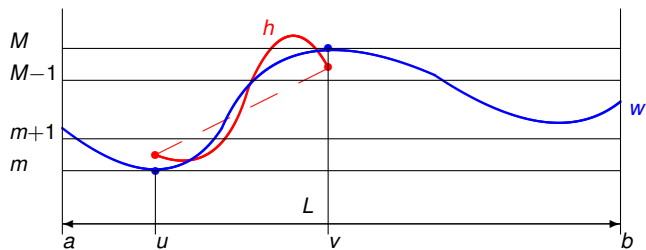
Subadditivity in time:

$$I_W^0(T_1 + T_2, r)^2 \leq I_W^0(T_1, r)^2 + I_{\widetilde{W}}^0(T_2, r)^2.$$

with independent W and \widetilde{W} .

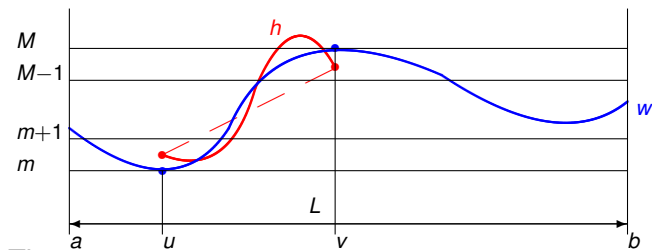
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Then

$$\int_a^b h'(t)^2 dt \geq \int_u^v h'(t)^2 dt \geq \frac{|h(u) - h(v)|^2}{|u - v|} \geq \frac{(M - m - 2)_+^2}{L}.$$

For Wiener process, $\frac{(M-m-2)_+^2}{L}$ scales to $(R - 2L^{-1/2})_+^2$ where R is the range of W on the unit interval.

Lower bound (continued)

By taking $T = nL$ and splitting $[0, T]$ into n intervals of length L we obtain

$$\frac{|h|_T^2}{T} \geq \frac{1}{nL} \sum_{j=1}^n (R^{(j)} - 2L^{-1/2})_+^2 \rightarrow L^{-1} \mathbb{E}(R - 2L^{-1/2})_+^2.$$

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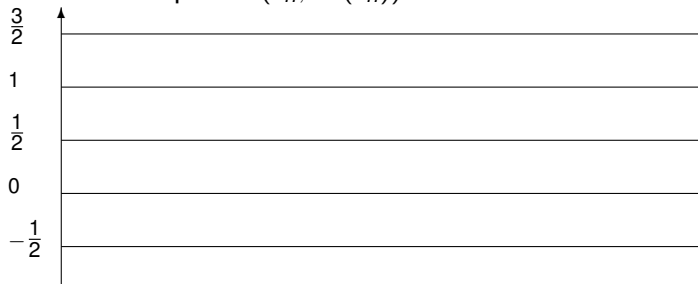
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$$c \geq 0.38$$

which is in agreement with empirical data.

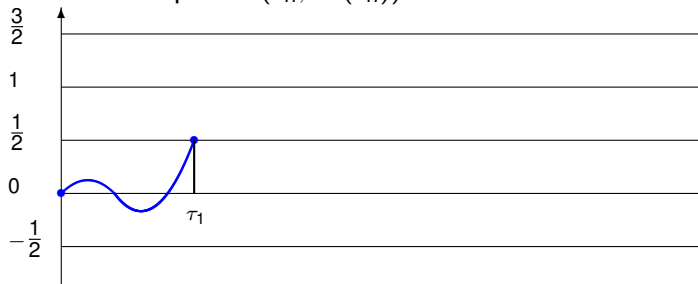
Upper bound: free-knot approximation

Let $\tau_{n+1} := \inf \{t \geq \tau_n \mid |W(t) - W(\tau_n)| \geq \frac{1}{2}\}$ Let $h(t)$ interpolate between the points $(\tau_n, W(\tau_n))$.



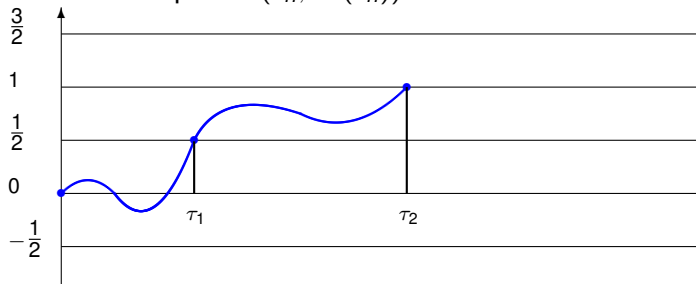
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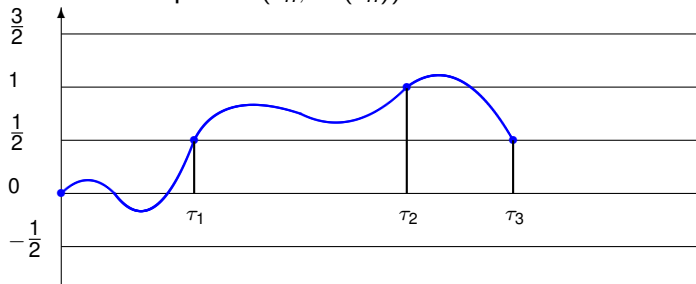
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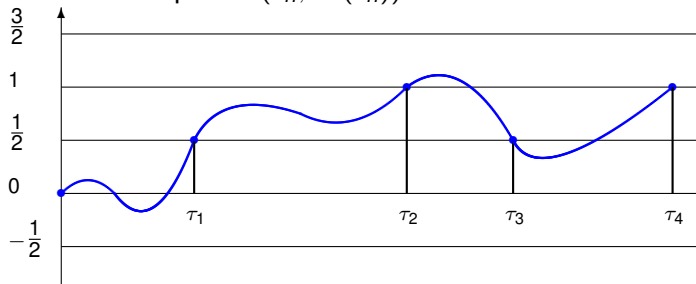
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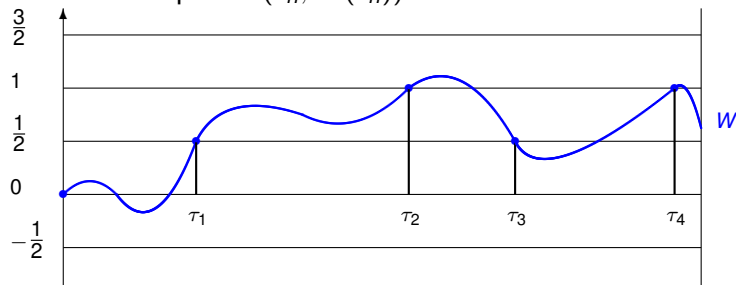
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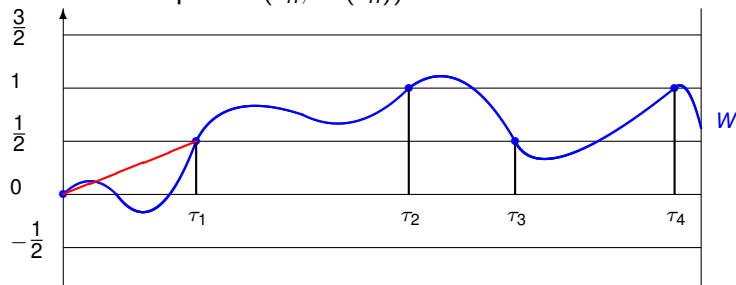
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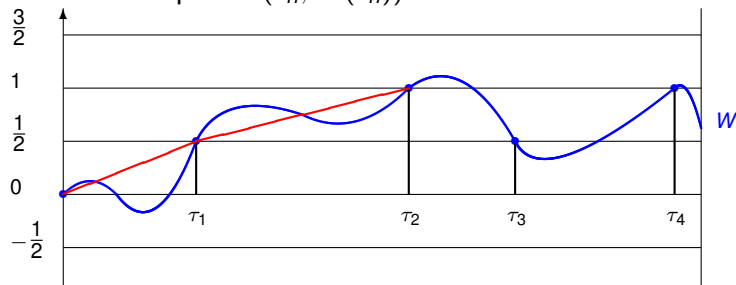
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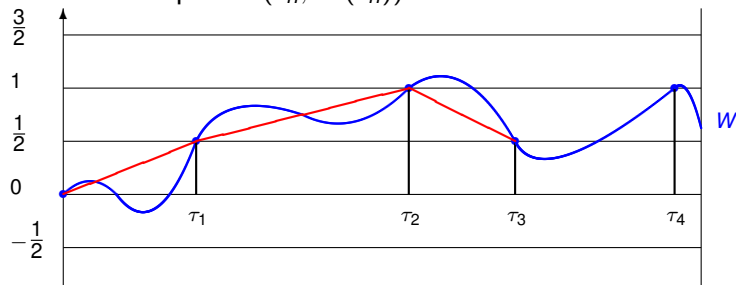
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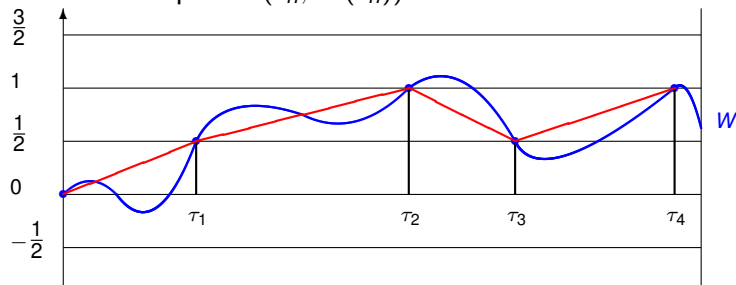
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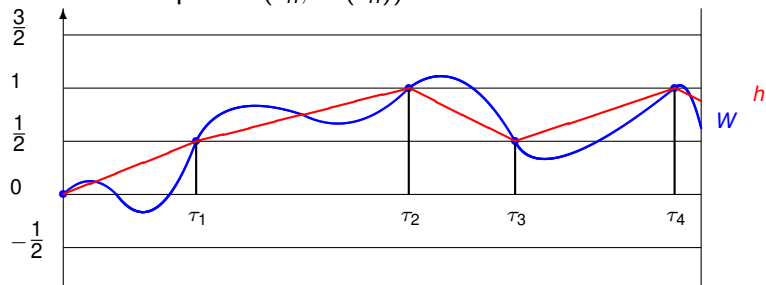
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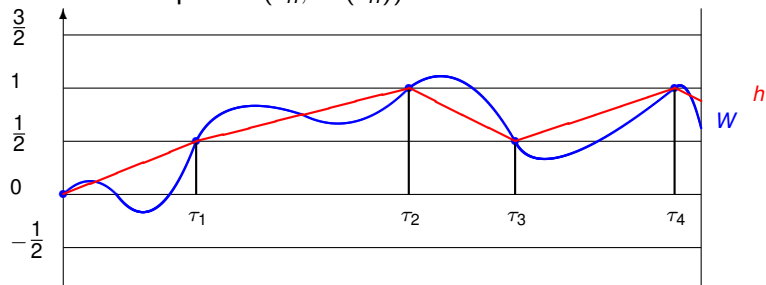
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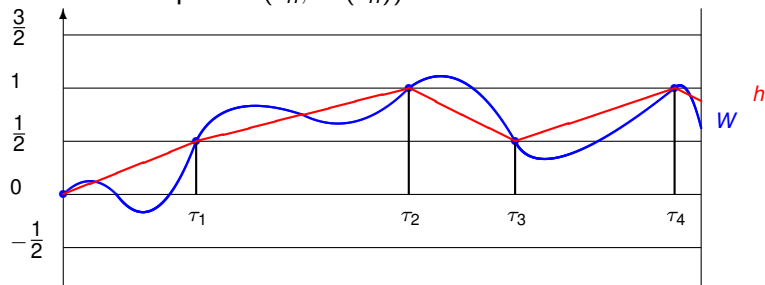
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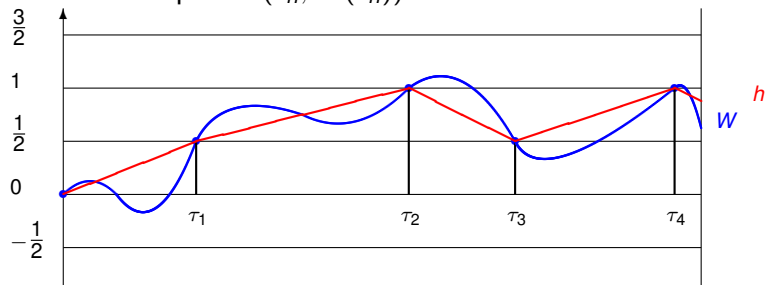
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Then $\forall t$ we have $|h(t) - W(t)| \leq 1$ and

$$\int_{\tau_n}^{\tau_{n+1}} h'(t)^2 dt = \frac{(h(\tau_{n+1}) - h(\tau_n))^2}{\tau_{n+1} - \tau_n} = \frac{1}{4(\tau_{n+1} - \tau_n)}$$

are i.i.d. random variables.

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Thus $C \leq 2\sqrt{1.832} \approx 2.7$.

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we see that

$$\mathbb{P}(W \notin \varepsilon U + c\varepsilon^{-1}K) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

whenever $c > \frac{\pi}{2}$.

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whenever $c > \frac{\pi}{2}$. It follows that $\mathcal{C} \leq \frac{\pi}{2}$.

Isoperimetric upper bound (Griffin–Kuelbs idea)

For any $c > 0, \varepsilon > 0$ we have

$$\mathbb{P}(\varepsilon I_W(1, \varepsilon \geq c)) = \mathbb{P}(W \notin \varepsilon U + c\varepsilon^{-1}K),$$

where $U := \{x : \|x\|_1 \leq 1\}$ and $K := \{h : |h|_1 \leq 1\}$.

By **Gaussian isoperimetric inequality**,

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whenever $c > \frac{\pi}{2}$. It follows that $\mathcal{C} \leq \frac{\pi}{2}$. This is the **best known** upper bound but it is **totally non-constructive**.

Markov pursuit: problem setting



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How to keep the Brownian dog on a leash in the energy saving mode?

Let the dog walk in \mathbb{R} according to a Brownian motion W .

You must follow it by moving with a finite speed and always stay not more than 1 away from the dog.

If $x(t)$ is your trajectory, then the goal is to follow the dog by expending minimal energy per unit of time

$$\frac{1}{T} \int_0^T x'(t)^2 dt$$

in a long run, $T \rightarrow \infty$.

Markov pursuit vs taut string

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The difference between the construction of pursuit and the taut string is huge: the former is built "online" based on the past and present trajectory of W while the latter requires the knowledge of entire trajectory of W .

Diffusion strategy for Markov pursuit

Let $X(t) := x(t) - W(t)$ be the signed distance to the dog.

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By ergodic theorem, in the stationary regime

$$\frac{1}{T} \int_0^T x'(t)^2 dt \rightarrow \int_{-1}^1 b(x)^2 p(x) dx = \int_{-1}^1 \frac{p'(x)^2}{p(x)^2} p(x) dx := I(p).$$

We have to **minimize Fisher information $I(p)$** !

Solution: optimal strategy for Markov pursuit

Minimizing Fisher information on the interval is a classical problem arising in Statistics, Data Analysis, etc (Zipkin, Huber, Levit, Shevlyakov, etc).

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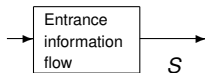
$$\frac{1}{T} \int_0^T x'(t)^2 dt \rightarrow I(p) = \pi^2.$$

We get

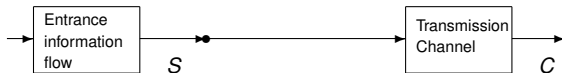
$$0.4 \approx C \leq I(p)^{1/2} = \pi.$$

This is a price to pay for not knowing the future.

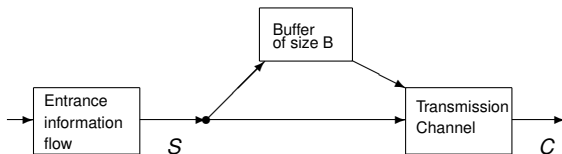
Related discrete applied problem



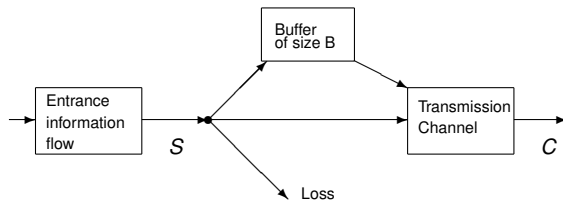
Related discrete applied problem



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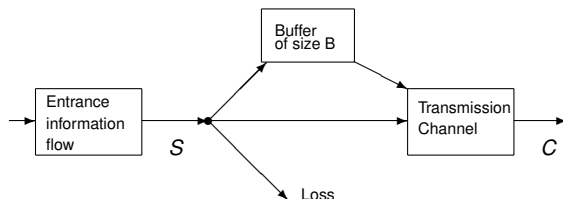


Related discrete applied problem



We have discrete time: $j = 1, 2, 3, \dots$

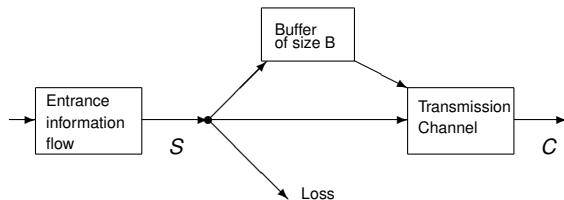
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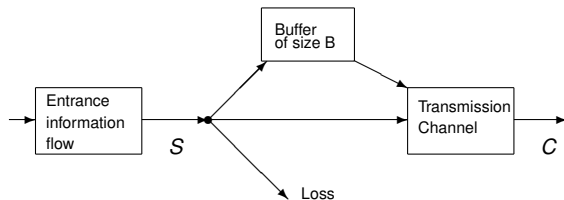


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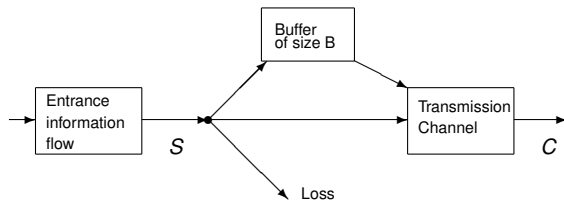
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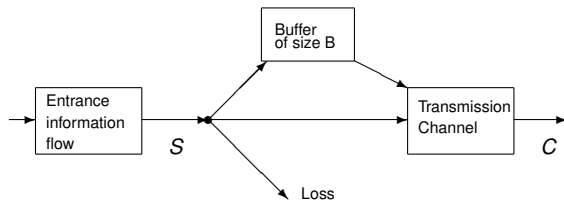
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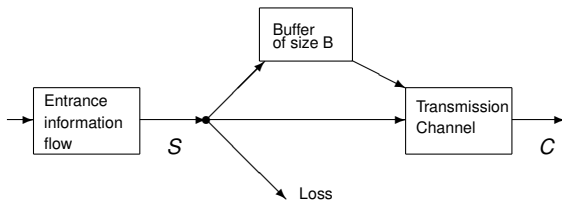
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Given $\varphi : [0, 1] \mapsto \mathbb{R}_+$ – increasing convex **penalty function**, define the **penalty functional**

$$F := \sum_{j=1}^n \varphi \left(\frac{L_j}{S_j} \right) S_j \searrow \min .$$

Buffer balance

We clearly have

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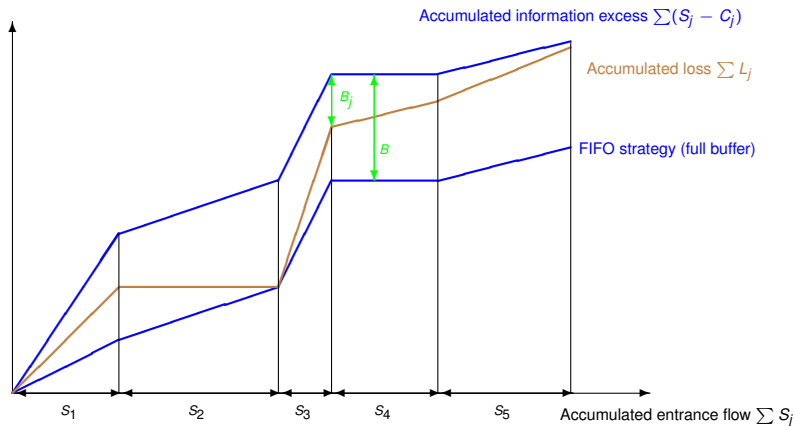
$$B_k = \sum_{j=1}^k (S_j - C_j) - \sum_{j=1}^k L_j.$$

Now buffer bounds $0 \leq B_k \leq B$ mean that

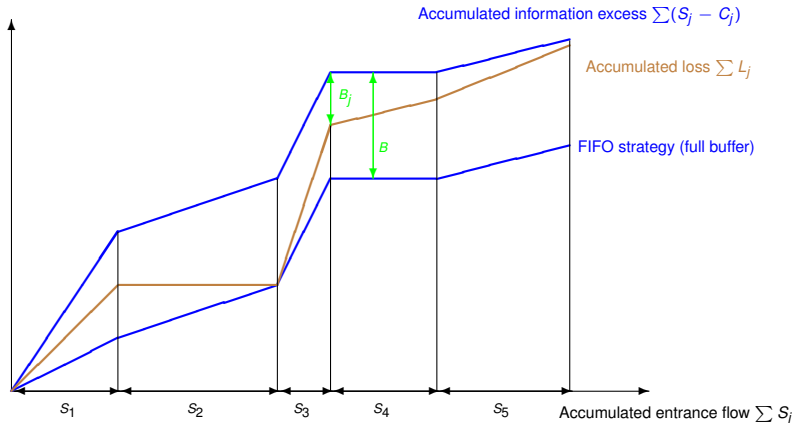
$$\sum_{j=1}^k (S_j - C_j) - B \leq \sum_{j=1}^k L_j \leq \sum_{j=1}^k (S_j - C_j).$$

That is $\sum_{j=1}^k L_j$ must go within a (random) band of fixed width B .

Buffer balance: graph

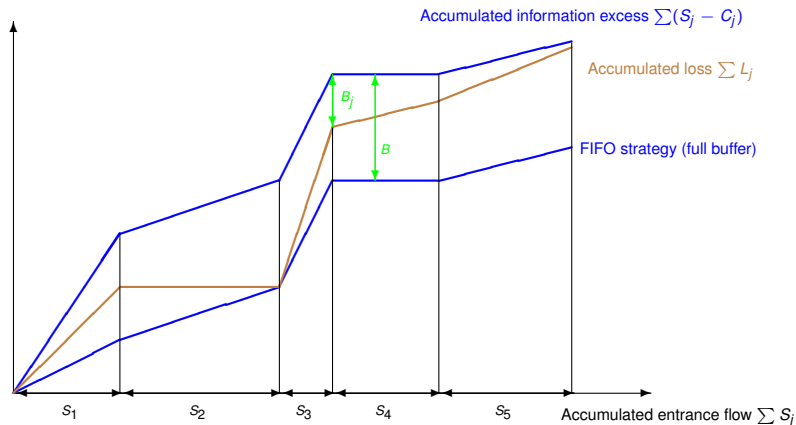


Buffer balance: graph



$$F = F(L) = \sum_{j=1}^n \varphi\left(\frac{L_j}{S_j}\right) S_j = \int_0^S \varphi(L') ds \rightarrow \min$$

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is attained on the taut string.

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Strassen's functional law of the iterated logarithm:

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etc.

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In terms of the taut string energy $I_W(\cdot, \cdot)$ we have

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etc. Here **the tube is much wider and the string energy is much lower** than in our case.