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Rearrangements and the Entropy Power Inequality

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Original Entropy Power Inequality

The Entropy Power Inequality (EPI) of [Shannon '48, Stam '59] can be formulated as follows:

EPI: Suppose X_1 and X_2 are two independent random vectors in $\mathbb{R}^n$ with finite differential entropies. Let Z_1 and Z_2 be two independent, isotropic (i.e., with covariance matrix a multiple of identity) Gaussian random vectors in $\mathbb{R}^n$ such that

$$h(X_1) = h(Z_1), \qquad h(X_2) = h(Z_2)$$

Then

$$h(X_1 + X_2) \geq h(Z_1 + Z_2)$$

Many motivations

- Implies, for i.i.d. X_i,

$$h\left(\frac{X_1 + X_2}{\sqrt{2}}\right) \geq h(X_1)$$

  and hence is a key step to the entropic Central Limit Theorem
- Useful in proving coding theorems in information theory
- Implies the Heisenberg Uncertainty Principle, and the Log Sobolev Inequality for Gaussian measures

A Comment and a Question

Comment: The functional $N(X) = e^{\frac{2h(X)}{n}}$ is called the "entropy power" of X. The EPI stated above is equivalent to the usual formulation: If X_1, X_2 are independent random vectors in $\mathbb{R}^n$,

$$N(X_1 + X_2) \geq N(X_1) + N(X_2)$$

with equality iff X_1 and X_2 are normal with proportional covariance matrices

Why? Let Z_1 and Z_2 be two independent, isotropic Gaussian random vectors in $\mathbb{R}^n$ such that

$$h(X_1) = h(Z_1), \qquad h(X_2) = h(Z_2)$$

Then

$$N(X_1) + N(X_2) = N(Z_1) + N(Z_2) = N(Z_1 + Z_2)$$

using the fact that for an isotropic Gaussian Z, $N(Z)$ is (up to an absolute constant) the variance of each component

Question: Can we refine the EPI, in the sense that we insert a meaningful quantity in between the two sides?

Reminder: What are Rearrangements?

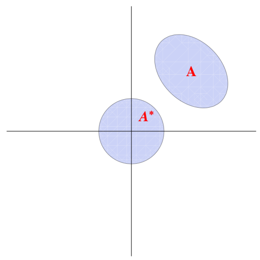
For a function $f : \mathbb{R}^n \to [0, \infty)$, let $A_t = \{x : f(x) \geq t\}$. Define

$$f^*(x) = \int_0^\infty 1_{\{x \in A_t{}^*\}} dt$$

where $1_{\{x \in A_t{}^*\}} = 1$ if $x \in A_t{}^*$ and 0 otherwise.

Remarks

- Note that $f(x) = \int_0^{f(x)} dt = \int_0^\infty 1_{\{f(x) \geq t\}} dt = \int_0^\infty 1_{\{x \in A_t\}} dt$
- The rearrangement of a set $A \subset \mathbb{R}^n$ is just the Euclidean ball with the same volume as A centered at 0, and is denoted A^*
- Rearrangements for sets $\subset \mathbb{R}^n$:



- So the idea of the definition is to build up f^* from the rearranged super-level sets in the same way that we can build f from its super-level sets

Some Lemmas

Lemma 1: For any function $f : \mathbb{R}^n \to [0, \infty)$, and all $t \geq 0$,

$$\{x : f(x) > t\}^* = \{x : f^*(x) > t\}$$

Lemma 2: (Rearrangement preserves densities) For any function $f : \mathbb{R}^n \to [0, \infty)$, and any $p \geq 1$,

$$\|f\|_p = \|f^*\|_p$$

In particular, the rearrangement of a density function is a density function

Lemma 3: (Rearrangement preserves entropy) For any density f,

$$h(f) = h(f^*)$$

Remarks

- Lemma 1 says that f^* is a spherically symmetric decreasing function (i.e., $f^*(x)$ only depends on $|x|$ and is non-increasing in it) such that, for any measurable subset $B \subset [0, \infty)$, the volumes of the sets $\{x : f(x) \in B\}$ and $\{x : f^*(x) \in B\}$ are the same
- Lemma 1 and Lemma 2 are classical; Lemma 3 appears to be a new observation but is not very difficult
- Basic idea of proofs: Tonelli's theorem

An entropy inequality with rearrangements

Theorem: Suppose X_1 and X_2 are two independent random vectors in $\mathbb{R}^n$ with densities f_1 and f_2, such that . Then, provided they exist,

$$h(X_1 + X_2) \geq h(X_1^* + X_2^*)$$

where X_1^* and X_2^* are independent with densities f_1^* and f_2^*

Remarks

- We will show that $h(X_i^*) = h(X_i)$. Hence, the EPI applied to X_1^* and X_2^* implies

  $$h(X_1^* + X_2^*) \geq h(Z_1 + Z_2)$$

  so that we can write

  $$h(X_1 + X_2) \geq h(X_1^* + X_2^*) \geq h(Z_1 + Z_2)$$

  Hence it can indeed be seen as a kind of strengthening
- However, note that this does not directly give a new proof of the EPI since we used the EPI to show that it was a strengthening!

Beyond Shannon entropy

Rényi entropy of order p: For $p > 1$,

$$h_p(X) = \frac{p}{p-1} \log \frac{1}{\|f\|_p}$$

where $\|f\|_p = \left(\int_{\mathbb{R}^n} f^p \, dx \right)^{1/p}$ is the usual L^p-norm on $\mathbb{R}^n$

The definition of $h_p(X)$ continues to make sense for $p \in (0,1)$ even though $\|f\|_p$ is then not a norm.

Special values of p: For $p = 0, 1, \infty$, $h_p(X)$ is defined "by continuity". In particular, as $p \to 1$, $h_p(X)$ reduces to the Shannon entropy

$$h(X) = h_1(X) = -\int_{\mathbb{R}^n} f(x) \log f(x) dx$$

Also

$$h_0(X) = \log |\text{Supp}(f)|,$$

where $\text{Supp}(f) =$ is the support of the density f, and

$$h_\infty(X) = -\log \|f\|_\infty$$

The General Results

Main Theorem: If X_i are independent $\mathbb{R}^n$-valued random vectors with densities f_i, and X_i^* are independent random vectors with densities f_i^*,

$$h_p(X_1 + X_2 + \ldots + X_k) \geq h_p(X_1^* + X_2^* + \ldots + X_k^*),$$

for any $p \in [0, \infty]$, provided both sides are well defined

Remarks

- Suppose $f_i, 1 \leq i \leq k$ are probability densities. Let $\phi(x)$ be a convex function defined on the non-negative real line such that $\phi(0) = 0$ and ϕ is continuous at 0. Then we actually show the even stronger result:

$$\int \phi(f_1 \star f_2 \star \cdots \star f_k(x))dx \leq \int \phi(f_1^* \star f_2^* \star \cdots \star f_k^*(x))dx,$$

provided that both sides are well defined

Proof ideas

For densities f and g on $\mathbb{R}^n$, we say that f is *majorized* by g if

$$\int_{\{x:\|x\|<r\}} f^*(x)dx \leq \int_{\{x:\|x\|<r\}} g^*(x)dx$$

for all $r > 0$. In this case, we write $f \prec g$

Fact: Let $\phi(x)$ be a convex function defined on $[0, \infty)$ such that $\phi(0) = 0$ and ϕ is continuous at 0. If f and g are densities with $f \prec g$, then

$$\int \phi(f(x))dx \leq \int \phi(g(x))dx,$$

provided that both sides are well defined

Remarks

- For non-negative ϕ, the Fact was proved by [Burchard '94]
- By taking $\phi(x) = x^p$ for $p > 1$, $\phi(x) = -x^p$ for $0 < p < 1$ and $\phi(x) = x\log(x)$ for $p = 1$, we recover the Main Theorem
- The fact that $f = f_1 \star f_2 \star \cdots \star f_k \prec g = f_1^* \star f_2^* \star \cdots \star f_k^*$ can be obtained as a consequence of the classical Rogers-Brascamp-Lieb-Luttinger inequalities for rearrangements [Rogers '57, Brascamp–Lieb–Luttinger '74]

The Unifying Character of The General Result

Brunn-Minkowski inequality

Let A, B be any Borel sets in $\mathbb{R}^n$. Write $A+B = \{x+y : x \in A,\ y \in B\}$ for the Minkowski sum, and $|A|$ for the n-dimensional volume. Then

$$\left|A + B\right|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}} \quad [BM]$$

Remarks

- BM follows from $p = 0$ case
- It has long been observed that BM resembles the EPI, and the two can be given a common proof via Young's inequality for convolution with a sharp constant [Lieb '78, Dembo–Cover–Thomas '91]
- $p = \infty$ case yields as a corollary an inequality due to Riesz and Sobolev

Implication for Fisher information inequalities

Recall the definition of Fisher information $I(f)$ for a density f:

$$I(f) = \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}$$

Corollary:

$$I(f) \geq I(f^*).$$

Remarks

- This turns out to be equivalent to the classical Pólya-Szegő inequality $\|\nabla f\|_p \geq \|\nabla f^*\|_p$ ($p \in [1, \infty]$) for $p = 2$, and gives a different proof of it
- Let g be an isotropic Gaussian density such that $h(g) = h(f)$. Then

$$I(f) \geq I(g) = \frac{1}{N(f)} \quad \text{[Stam '59]}$$

  is the "isoperimetric inequality for entropy". Just as our Main Theorem strengthens the EPI, the above strengthens the isoperimetric inequality for entropy since it inserts $I(f^*)$ in between $I(f)$ and $I(g)$

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$\surd$ Background: EPI; Rearrangements

$\surd$ An entropy inequality involving rearrangements

$\surd$ General formulations and proof ideas

$\surd$ The unifying character of the general result

- A conjecture involving generalized Gaussians
- A new approach to the classical EPI
- Discrete analogues

Towards a Rényi EPI

The *Rényi entropy power* of order p is $N_p(X) = \exp\{\frac{2h_p(X)}{n}\}$

Theorem:[Bobkov–Chistyakov '13] If $X_1, \ldots, X_k$ are independent random vectors taking values in $\mathbb{R}^n$, then for any $p \geq 1$,

$$N_p(X_1 + \ldots + X_k) \geq c_p \sum_{i=1}^{k} N_p(X_i),$$

where c_p is a constant depending only on p

Remarks

- For $p \in (1, \infty)$, [Bobkov–Chistyakov '13] showed that $c_p = \frac{1}{e} p^{\frac{1}{p-1}}$ works
- The $p = 1$ case is simply the EPI, which is sharp for Gaussians with proportional covariance matrices
- For $p = 1$, [Rogozin '87] showed that $c_\infty = \frac{1}{2}$ is sharp when $n = 1$, while [Bobkov–Chistyakov '12] showed that $c_\infty = \frac{1}{2}$ is sharp when $k = 2$
- In general, the theorem is not sharp

Generalized Gaussians

For $-\infty < \beta \le \frac{2}{n+2}$, the *standard generalized Gaussian* of order β has density

$$g_\beta(x) = A_\beta \left(1 - \frac{\beta}{2}\|x\|^2\right)_+^{\frac{1}{\beta}-\frac{n}{2}-1},$$

where A_β is a normalizing constant

Remarks

- g_0 is the standard Gaussian density in $\mathbb{R}^n$
- For $\beta < 0$, g_β is proportional to a negative power of $(1 + b\|x\|^2)$ for a positive constant b, and therefore correspond to heavy-tailed measures with full support on $\mathbb{R}^n$ (Student-t distributions)
- For $0 < \beta \le \frac{2}{n+2}$, g_β is a concave function supported on a centered Euclidean ball of finite radius (Student-r distributions)
- Although the first class includes many distributions from what one might call the "Cauchy family", it excludes the standard Cauchy distribution. Indeed, the form of g_β has been chosen so that, for $Z \sim g_\beta$, $E[\|Z\|^2] = n$ for any β

A conjecture involving generalized Gaussians

$$\text{For } p > \frac{n}{n+2}, \quad \text{define} \quad \frac{1}{\beta_p} = \frac{1}{p-1} + \frac{n+2}{2}$$

Fact:[Costa–Hero–Vignat '03, Lutwak–Yang–Zhang '07] If X is a random vector taking values in $\mathbb{R}^n$, then for any $p > \frac{n}{n+2}$,

$$\frac{E[\|X\|^2]}{N_p(X)} \geq \frac{E[\|Z^{(p)}\|^2]}{N_p(Z^{(p)})},$$

where $Z^{(p)} \sim g_{\beta_p}$

Conjecture: Let $X_1, \ldots, X_k$ be independent random vectors taking values in $\mathbb{R}^n$, and $p > \frac{n}{n+2}$. Suppose Z_i are independent random vectors, each a scaled version of $Z^{(p)}$. such that $h_p(X_i) = h_p(Z_i)$. Then

$$h_p(X_1 + \ldots + X_k) \geq h_p(Z_1 + \ldots + Z_k).$$

Remarks

- The conjecture suggests optimal constants for the Rényi EPI in general
- It is true for the three special cases where sharp constant is known
- Our main result "refines" this conjecture

A New Proof of the EPI: IID case

Can we get a new proof of the Original EPI from our Main Result?

Folklore proof for symmetric densities

By the scaling property for entropy, and by subadditivity,

$$h(Y_1, Y_2) \ = \ h\left(\frac{Y_1 + Y_2}{\sqrt{2}}, \frac{Y_1 - Y_2}{\sqrt{2}}\right) \ \leq \ h\left(\frac{Y_1 + Y_2}{\sqrt{2}}\right) + h\left(\frac{Y_1 - Y_2}{\sqrt{2}}\right)$$

But, by independence, $h(Y_1, Y_2) = h(Y_1) + h(Y_2)$, and by spherical symmetry (in fact, we only need central symmetry) and i.i.d. assumption, we have

$$Y_1 + Y_2 \ =^{\mathcal{D}} \ Y_1 - Y_2.$$

Hence

$$h\left(\frac{Y_1 + Y_2}{\sqrt{2}}\right) \geq h(Y_1) \tag{1}$$

Remarks

- The Main Result implies $h(Y_1 + Y_2) \geq h(Y_1^* + Y_2^*)$, which combined with the folklore observation, yields (1) for all i.i.d. Y_1, Y_2
- There are many proofs of EPI [Stam '59, Blachman '65, Lieb '78, Szarek–Voiculescu '00, Verdú–Guo '06, Rioul '11]

A New Proof of the EPI: general case

Our goal is to recover, from our Main Theorem, the following formulation of EPI: for any $0 < \lambda < 1$,

$$h(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) \geq \lambda h(X) + (1-\lambda)h(Y)$$

where, for notational simplicity, we will only consider $n = 1$

Three Reductions of the Problem

1. By the Main Theorem, we can assume that X and Y are symmetric, unimodal random variables
2. By the tensorization trick, it is sufficient to show

$$h(\sqrt{\lambda}\mathbf{X} + \sqrt{1-\lambda}\mathbf{Y}) \geq \lambda h(\mathbf{X}) + (1-\lambda)h(\mathbf{Y}) + o(M)$$

   where $\mathbf{X}$ and $\mathbf{Y}$ consist of M independent copies of X and Y respectively
3. Suppose $X \sim f$ and $Y \sim g$. Standard approximation arguments allow us to take f and g to be simple functions, i.e., mixtures of uniform distributions on symmetric intervals. Then:
   - densities of $\mathbf{X}$ and $\mathbf{Y}$ are simple functions
   - densities of $\mathbf{X}^*$ and $\mathbf{Y}^*$ are mixtures of uniforms on balls

Concavity Properties of Entropy

If $p = \sum_{i=1}^{r} c_i f_i$ is a mixture of densities, then

$$\sum_{i=1}^{r} c_i h(f_i) \le h(p) \le \sum_{i=1}^{r} c_i h(f_i) + \log(r)$$

Remarks

- The right inequality– a "reverse concavity" of the entropy functional– is very easy but seems to be a new observation
- Since a convolution of mixtures is a mixture of convolutions, the above concavity properties can be used to reduce the EPI involving $\mathbf{X}$ and $\mathbf{Y}$ (from previous slide) to just an EPI involving uniforms on balls
- By comparison with Gaussians and a use of Stirling's formula, we can show that

$$h(\sqrt{\lambda}\mathbf{Z_1} + \sqrt{1-\lambda}\mathbf{Z_2}) \ge \lambda h(\mathbf{Z_1}) + (1-\lambda)h(\mathbf{Z_2}) - C\log(M)$$

  where C is a universal constant and $\mathbf{Z_1}, \mathbf{Z_2}$ are uniforms on balls of possibly different radii

Summary

- We can refine the EPI using rearrangements. In fact, such an inequality (the Main Theorem) extends to Rényi entropies of any order $p \in [0, \infty]$
- EPI can be obtained as a consequence of the Main Theorem
- The Main Theorem gives a clean unification of EPI and Brunn-Minkowski inequality (and also of some other inequalities)
- There is an analogue on the integers, which has connections to additive combinatorics