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STRONG APPROXIMATIONS TO A QUANTILE PROCESS BASED ON INDEPENDENT FRACTIONAL BROWNIAN MOTIONS

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This is joint work with Péter Kevei

ABSTRACT

Swanson (2007) using classical weak convergence theory proved that an appropriately scaled median of n independent Brownian motions converges weakly to a mean zero Gaussian process. More recently Kuelbs and Zinn (2013) have obtained central limit theorems for a quantile process based n independent copies of certain random processes. These include fractional Brownian motions, which may be zero or perturbed to be not zero with probability 1 at zero. Their approach is based on an extension of a result of Vervaat (1972) on the weak convergence of inverse processes. We shall define a time dependent empirical process based n independent fractional Brownian motions and discuss strong approximations to it by Gaussian processes. We follow the basic methodology outlined in Berthet and Mason (2006) to obtain our approximations. Surprisingly, these approximations are in force on sequences of intervals for which weak convergence cannot hold in the limit. They lead to strong approximations and functional laws of the iterated logarithm for the quantile or inverse of these empirical processes and are obtained via Bahadur-Kiefer representations. May talk is based on joint work with Péter Kevei.

SWANSON RESULT

Let $\left\{B_j^{(1/2)}\right\}_{j\geq 1}$ be a sequence of i.i.d. standard Brownian motions and for each $n \geq 1$ and $t \geq 0$ let $M_n(t)$ denote the median of

$$B_1^{(1/2)}(t), \ldots, B_n^{(1/2)}(t).$$

Swanson (2007) using classical weak convergence theory proved that $\sqrt{n}M_n(t)$ converges weakly to a continuous centered Gaussian process X on $[0, \infty)$ with covariance function defined for $t_1, t_2 \in [0, \infty)$ by

$$\mathbb{E}\left(X(t_1)\, X(t_2)\right) = \sqrt{t_1 t_2}\sin^{-1}\left(\frac{t_1 \wedge t_2}{\sqrt{t_1 t_2}}\right).$$

The aim of this talk is to place this result within the context of what has long known about the usual empirical and quantile processes.

CLASSICAL QUANTILE PROCESS LORE

Let $X_1, X_2, \ldots,$ be i.i.d. F. For $\alpha \in (0,1)$ define the *inverse* or *quantile function*

$$Q(\alpha) = \inf\{x : F(x) \geq \alpha\},$$

and the *empirical inverse* or *empirical quantile function*

$$Q_n(\alpha) = \inf\{x : F_n(x) \geq \alpha\},$$

where

$$F_n(x) = n^{-1} \sum_{j=1}^{n} 1\{X_j \leq x\},\ x \in \mathbb{R}.$$

is the empirical distribution based on $X_1, \ldots, X_n$.

EMPIRICAL AND QUANTILE PROCESSES

We define the empirical process

$$v_n(x) := \sqrt{n}\left\{F_n(x) - F(x)\right\},\ x \in \mathbb{R}$$

and the quantile process

$$u_n(t) := \sqrt{n}\left\{Q_n(t) - Q(t)\right\},\ t \in (0,1).$$

BAHADUR-KIEFER REPRESENTATION

Let $X_1, X_2, \ldots,$ be i.i.d. F on $[0,1]$, where F is twice differentiable on $(0,1)$, with

$$\inf_{x\in(0,1)} f(x) = F'(x) > 0 \text{ and } \sup_{x\in(0,1)} |F''(x)| < \infty.$$

We have (Kiefer (1970))

$$\limsup_{n\to\infty} \frac{n^{1/4} \|v_n(Q) + f(Q)\, u_n\|_{(0,1)}}{\sqrt[4]{\log\log n}\sqrt{\log n}} = \frac{1}{\sqrt[4]{2}}, \text{ a.s.}$$

(The function $f(Q)$ is called the density quantile function.) Deheuvels and Mason (1990) developed a general approach to such theorems.

SOME NOTATION

For a real-valued function Υ defined on a set S we shall use the notation

$$\|\Upsilon\|_S = \sup_{s \in S} |\Upsilon(s)| .$$

In particular,

$$\|v_n(Q) + f(Q) u_n\|_{(0,1)}$$

$$= \sup_{t \in (0,1)} |v_n(Q(t)) + f(Q(t)) u_n(t)| .$$

KMT STRONG APPROXIMATION

Using a strong approximation result of KMT (1975) one has on the same probability space an i.i.d. F sequence $X_1, X_2, \ldots,$ and a sequence of i.i.d. Brownian bridges $B_1, B_2, \ldots,$ such that

$$\left\| v_n(Q) - \frac{\sum_{j=1}^{n} B_j}{\sqrt{n}} \right\|_{(0,1)} = O\left(\frac{(\log n)^2}{\sqrt{n}} \right), \text{ a.s.}$$

Under the conditions for which the above B-K representation holds

$$\limsup_{n\to\infty} \frac{n^{1/4} \left\| \frac{\sum_{j=1}^{n} B_j}{\sqrt{n}} + f(Q)u_n \right\|_{(0,1)}}{\sqrt[4]{\log\log n}\sqrt{\log n}} = \frac{1}{\sqrt[4]{2}}, \text{ a.s.}$$

Deheuvels (1998) has shown that this rate of strong approximation rate cannot be improved.

MAIN FOCUS OF OUR TALK

We shall discuss analogs of these classical results for time dependent empirical and quantile processes based on independent copies of fractional Brownian motion.

This will put the Swanson result into a larger context.

FRACTIONAL BROWNIAN MOTION

Let

$$\left\{B^{(H)}\right\} \cup \left\{B_j^{(H)}\right\}_{j\geq 1}$$

be a sequence of i.i.d. fractional Brownian motions with Hurst index $0 < H < 1$ defined on $[0, \infty)$.

Note that $B^{(H)}$ is a continuous mean zero Gaussian process on $[0, \infty)$ with covariance function defined for any s,$t \in [0, \infty)$

$$\mathbb{E}\left(B^{(H)}(s)\, B^{(H)}(t)\right) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right).$$

A TIME DEPENDENT EMPIRICAL DISTRIBUTION

For any $t \in [0, \infty)$ and $x \in \mathbb{R}$ let

$$F_H(t, x) = P\left\{B^{(H)}(t) \le x\right\}$$

and for $n \ge 1$ define the time dependent *empirical distribution function*

$$F_n(t, x) = n^{-1} \sum_{j=1}^{n} 1\left\{B_j^{(H)}(t) \le x\right\}.$$

Note that

$$F_H(t, x) = P\left\{B^{(H)}(t) \le x\right\} = \Phi\left(x/t^H\right),$$

where

$$\Phi(x) = P\{Z \le x\},$$

with Z being a standard normal random variable.

A TIME DEPENDENT EMPIRICAL PROCESS

Define the time dependent *empirical process* indexed by $(t, x) \in \mathcal{T}(\gamma)$,

$$\nu_n(t, x) = \sqrt{n}\left\{F_n(t, x) - F_H(t, x)\right\}$$

where

$$\mathcal{T}(\gamma) := [\gamma, T] \times \mathbb{R}, \text{ with } 0 < \gamma \leq 1 < T < \infty.$$

WEAK CONVERGENCE

Applying Theorem 5 of Kuelbs, Kurtz and Zinn (2012) one can show that $\nu_n(t,x)$ converges weakly to a uniformly continuous centered Gaussian process $G(s,x)$ indexed by $(s,x) \in \mathcal{T}(\gamma)$, whose trajectories are bounded, having covariance function

$$\mathbb{E}\left(G(s,x)\,G(t,y)\right) = P\left\{B^{(H)}(s) \le x, B^{(H)}(t) \le y\right\}$$
$$-P\left\{B^{(H)}(s) \le x\right\} P\left\{B^{(H)}(t) \le y\right\}.$$

A GAUSSIAN PROCESS

It will be convenient to let $G_{(\gamma,T)}$ denote the mean zero Gaussian process indexed by $\mathcal{T}(\gamma)$, having covariance function defined for

$$(s,x),(t,y) \in \mathcal{T}(\gamma)$$

by

$$\mathbb{E}\left(G_{(\gamma,T)}(s,x)\,G_{(\gamma,T)}(t,y)\right) = \mathbb{E}\left(G(s,x)\,G(t,y)\right).$$

PROBABILISTICALLY EQUIVALENT VERSION

We shall say that a process $\widetilde{\mathcal{Y}}$ is a *probabilistically equivalent version* of $\mathcal{Y}$ if

$$\widetilde{\mathcal{Y}} \stackrel{\mathrm{D}}{=} \mathcal{Y}.$$

A GAUSSIAN COUPLING

In the results that follow $\nu_0 = 2 + \frac{2}{H}$ and $H_0 = 1 + H$. As long as $1 \geq \gamma = \gamma_n > 0$ satisfies the ETA Condition, namely, for some $\frac{1}{5H_0} > \eta \geq 0$,

$$\infty > -\log \gamma_n / \log n \to \eta, \text{ as } n \to \infty,$$

then for every $\lambda > 1$ there exists a $\rho(\lambda) > 0$ such that for all n large enough one can construct on the same probability space random vectors $B_1^{(H)}, \ldots, B_n^{(H)}$ i.i.d. $B^{(H)}$ and a probabilistically equivalent version $\widetilde{G}^{(n)}_{(\gamma_n,T)}$ of $G_{(\gamma_n,T)}$ such that with $\tau_2 = \tau_2(H) > 0$

$$\mathbb{P}\left\{ \left\| v_n - \widetilde{G}^{(n)}_{(\gamma_n,T)} \right\|_{\mathcal{T}(\gamma_n)} > \frac{\rho(\lambda)\,(\log n)^{\tau_2}}{\left(n^{1/2}\gamma_n^{5H/2}\right)^{2/(2+5\nu_0)}} \right\} \leq n^{-\lambda}.$$

SPECIAL CASE

In particular, when $\gamma_n = n^{-\eta}$, with $0 \leq \eta < \frac{1}{5H_0}$,

$$\mathbb{P}\left\{ \left\| \nu_n - \widetilde{G}^{(n)}_{(\gamma_n,T)} \right\|_{\mathcal{T}(\gamma_n)} > \rho\,(\lambda)\, n^{-\tau_1}\,(\log n)^{\tau_2} \right\} \leq n^{-\lambda},$$

where $\tau_1 = \tau_1(\eta) = (1 - 5H_0\eta)\,/\,(2 + 5\nu_0) > 0.$

UNEXPECTED RESULT

This is quite unexpected in light of the results in Kuelbs, Kurtz and Zinn (2012). They point out that the empirical process $\nu_n(t,x)$ indexed by $[0,T]\times\mathbb{R}$ does not converge weakly to a uniformly continuous centered Gaussian process indexed by $(t,x)\in[0,T]\times\mathbb{R}$ whose trajectories are bounded.

However, our coupling implies that on a suitable probability space

$$\left\|\nu_n - \widetilde{G}_{(\gamma_n,T)}\right\|_{\mathcal{T}(\gamma_n)} \xrightarrow{\text{P}} 0, \text{ as } n \to \infty,$$

where γ_n goes to zero.

A STRONG APPROXIMATION

Our coupling leads to the following Kiefer process type strong approximation result: as long as $1 \geq \gamma = \gamma_n > 0$ is constant, $1/(2\tau_1(0)) < \alpha < 1/\tau_1(0)$ and $\xi > 1$ there exist a sequence of i.i.d. $B_1^{(H)}, B_2^{(H)}, \ldots,$ and a sequence of independent copies $G_{(\gamma,T)}^{(1)}, G_{(\gamma,T)}^{(2)}, \ldots$, of $G_{(\gamma,T)}$ sitting on the same probability space such that a.s.

$$\max_{1 \leq m \leq n} \left\| \sqrt{m} v_m - \sum_{i=1}^{m} G_{(\gamma,T)}^{(i)} \right\|_{\mathcal{T}(\gamma)} = O\left(n^{1/2 - \tau(\alpha)} \left(\log n\right)^{\tau_2} \right).$$

where $\tau(\alpha) = (\alpha\tau_1(0) - 1/2)/(1+\alpha) > 0$ and $\tau_1(0) = 1/(2 + 5\nu_0)$.

APPLICATIONS TO COMPACT LIL

Our strong approximation obviously implies that for any fixed choice of $0 < \gamma \le 1 < T$ there exist on the same probability space a sequence of i.i.d. $B_1^{(H)}, B_2^{(H)}, \ldots,$ fractional Brownian motions on $[\gamma, T]$ with Hurst index $0 < H < 1$ and a sequence of independent copies $G_{(\gamma,T)}^{(1)}, G_{(\gamma,T)}^{(2)}$, $\ldots$, of $G_{(\gamma,T)}$ such that a.s.

$$\max_{1\le m\le n} \left\| \sqrt{m} v_m - \sum_{i=1}^{m} G_{(\gamma,T)}^{(i)} \right\|_{\mathcal{T}(\gamma)} = o\left(\sqrt{n \log\log n}\right).$$

Applying Corollary 2.2 of Arcones (1995) we obtain a compact law of the iterated logarithm (LIL) for

$$n^{-1/2} \sum_{i=1}^{n} G_{(\gamma,T)}^{(i)},$$

which immediately also holds for v_n.

IN PARTICULAR WE GET

In particular we get that

$$\limsup_{n\to\infty} \frac{\|v_n\|_{\mathcal{T}(\gamma)}}{\sqrt{2\log\log n}} =$$

$$\limsup_{n\to\infty} \sup_{(t,x)\in\mathcal{T}(\gamma)} \left| \frac{\nu_n(t,x)}{\sqrt{2\log\log n}} \right| = \sigma(\gamma,T), \text{ a.s.}$$

where

$$\sigma^2(\gamma,T) = \sup\left\{ \mathbb{E}\left(G^2_{(\gamma,T)}(t,x) \right) : (t,x) \in \mathcal{T}(\gamma) \right\}.$$

A TIME DEPENDENT QUANTILE PROCESS

For each $t \in (0, \infty)$ and $\alpha \in (0, 1)$ define the *inverse* or *quantile function*

$$\tau_\alpha(t) = \inf\{x : F_H(t, x) \geq \alpha\},$$

and the time dependent *empirical inverse* or *empirical quantile function*

$$\tau_\alpha^n(t) = \inf\{x : F_n(t, x) \geq \alpha\},$$

and the corresponding time dependent *quantile process*

$$u_n(t, \alpha) := \sqrt{n}\{\tau_\alpha^n(t) - \tau_\alpha(t)\}.$$

SOME OBSERVATIONS

Notice that for each fixed $t > 0$, $F_H(t, x)$ has density

$$f_H(t, x) = \frac{1}{t^H \sqrt{2\pi}} \exp\left(-\frac{x^2}{2t^{2H}}\right), \ -\infty < x < \infty.$$

Further, for each $t \in (0, \infty)$ and $\alpha \in (0, 1)$, $\tau_\alpha(t)$ is uniquely defined by

$$\tau_\alpha(t) = t^H z_\alpha, \text{ where } P\{Z \leq z_\alpha\} = \alpha,$$

which says that the time dependent density quantile is

$$f_H(t, \tau_\alpha(t)) = \frac{1}{t^H \sqrt{2\pi}} \exp\left(-\frac{z_\alpha^2}{2}\right).$$

Notice that

$$u_n(t, 1/2) := \sqrt{n}\tau_{1/2}^n(t) = \sqrt{n} M_n(t),$$

is the Swanson median process in the special case when

$$B^{(H)} = B^{(1/2)}$$

is Brownian motion.

WHAT IS KNOWN

Besides the Swanson result, the following is known. Kuelbs and Zinn (2013) consider the sequence of quantile processes $\left\{u_n^Y\right\}_{n\geq 1}$ based on i.i.d. self-similar processes $\left\{Y_j\right\}_{j\geq 1}$.

In this setup they prove using an extension of a result of Vervaat (1972) on the weak convergence of inverse processes that for any $0 < \rho < 1/2$ and $T > 1$, the process

$$u_n^Y\left(t, \alpha\right)$$

converges weakly to a mean zero continuous Gaussian process on $[0, T] \times [\rho, 1 - \rho]$. Their result implies the Swanson result.

A B-K KIEFER REPRESENTATION

Whenever $0 < \gamma = \gamma_n \leq 1$ satisfies the above ETA condition for some $\frac{1}{5H_0} > \eta \geq 0$, then for any $0 < \rho < 1/2$ and $H > \delta > 0$

$$\sup_{(t,\alpha)\in[\gamma_n,T]\times[\rho,1-\rho]} \left| \nu_n \left(t, \tau_\alpha \left(t\right)\right) + \frac{\exp\left(-\frac{z_\alpha^2}{2}\right)}{t^H\sqrt{2\pi}} u_n\left(t,\alpha\right) \right|$$

$$= O\left(n^{-1/4}\gamma_n^{-H/2-\delta}\left(\log\log n\right)^{1/4}\left(\log n\right)^{1/2}\right), \text{ a.s.}$$

It is noteworthy here to point out that when $\gamma_n = \gamma$ is constant the rate we get corresponds to the exact rate

$$n^{-1/4}\left(\log\log n\right)^{1/4}\left(\log n\right)^{1/2}$$

in the classic uniform Bahadur-Kiefer representation

IMPLICATION (I)

Notice when $0 < \gamma \leq 1$ is fixed, we immediately get from the compact LIL for v_n that

$$\left\{ \frac{f_H\left(t, \tau_\alpha\left(t\right)\right) u_n\left(t, \alpha\right)}{\sqrt{2 \log \log n}} : \left(t, \alpha\right) \in [\gamma, T] \times [\rho, 1-\rho] \right\}$$

is relatively compact in the class of bounded functions on $[\gamma, T] \times [\rho, 1-\rho]$ and its limit set is the unit ball of the reproducing kernel Hilbert space determined by the covariance function defined for $(t_1, \alpha_1), (t_2, \alpha_2) \in [\gamma, T] \times [\rho, 1-\rho]$ by

$$K\left(\left(t_1, \alpha_1\right), \left(t_2, \alpha_2\right)\right) = \mathbb{E}\left(G\left(t_1, \tau_{\alpha_1}\left(t_1\right)\right) G\left(t_2, \tau_{\alpha_2}\left(t_2\right)\right)\right)$$

$$= P\left\{ B^{(H)}\left(t_1\right) \leq t^H z_{\alpha_1}, B^{(H)}\left(t_2\right) \leq t^H z_{\alpha_2} \right\} - \alpha_1 \alpha_2$$

IMPLICATION (II)

Also we get when $0 < \gamma \leq 1$ is fixed the following Kiefer-type strong approximation,

$$\sup_{(t,\alpha)\in[\gamma,T]\times[\rho,1-\rho]} \left| \sqrt{n} f_H\left(t, \tau_\alpha\left(t\right)\right) u_n\left(t,\alpha\right) + \sum_{i=1}^{n} G_i\left(t, \tau_\alpha\left(t\right)\right) \right|$$

$$= O\left(n^{1/2-\tau(\alpha)}\left(\log n\right)^{\tau_2}\right), \text{ a.s.,}$$

where $\tau\left(\alpha\right) < 1/4$.

RETURNING TO SWANSON

Implications I and II eventually lead to the following LIL and strong approximation for Swanson's median process. When $H = 1/2$,

$$\limsup_{n\to\infty} \frac{\left\|\sqrt{n}\tau^n_{1/2}\right\|_{[0,T]}}{\sqrt{2\log\log n}} = \sqrt{T\pi/2}, \text{ a.s.},$$

and there exist a sequence $B_1^{(1/2)}, B_2^{(1/2)}, \ldots$, i.i.d. $B^{(1/2)}$ and a sequence of processes $X^{(1)}, X^{(2)}, \ldots$, i.i.d. X sitting on the same probability space such that, a.s.

$$\left\|\sqrt{n}\tau^n_{1/2} - \frac{1}{\sqrt{n}}\sum_{i=1}^{n} X^{(i)}\right\|_{[0,T]} = o\left(1\right),$$

which follows from the fact that $X\left(t\right) \overset{\mathrm{D}}{=} t^{1/2}B^{(1/2)}(t)$. Of course, this implies the Swanson result that $\sqrt{n}\tau^n_{1/2}$ converges weakly to the process X.

INGREDIENTS OF STRONG APPROXIMATION PROOFS

We essentially followed a procedure to establish strong approximations to empirical processes indexed by classes of functions given in Berthet and Mason (2006), which, in turn, was motivated by the work of Dudley and Philipp (1983). The ingredients include exponential inequalities for empirical and Gaussian processes, moment bounds for the supremum of the oscillation of these processes and good bounds for the entropy of certain indexing sets.

The most important tool was a coupling result based on a Gaussian approximation of Zaitsv (1987).

COUPLING BASED ON ZAITSEV (1987)

Let $Y_1, \ldots, Y_n$ be independent mean zero random vectors on R^N, $N \geq 1$, such that for some $B > 0$,

$$|Y_i|_N \leq B, \ i = 1, \ldots, n.$$

If (Ω, T, P) is rich enough then for each $\delta > 0$, one can define independent normally distributed mean zero random vectors $Z_1, \ldots, Z_n$ with

$$cov\,(Z_i) = cov\,(Y_i)\,,$$

for $i = 1, \ldots, n$, such that for universal constants $C_1 > 0$ and $C_2 > 0$,

$$\mathbb{P}\left\{\left|\sum_{i=1}^{n}(Y_i - Z_i)\right|_N > \delta\right\} \leq C_1 N^2 \exp\left(-\frac{C_2\delta}{N^2 B}\right).$$

This result was pointed out in Einmahl and Mason (1997).

LANDAU AND SHEPP (1970) THEOREM

The Lévy modulus of continuity theorem for fractional Brownian motion $B^{(H)}$ with Hurst index $0 < H < 1$ (Wang (2001)), says that for any $0 < T < \infty$, with probability 1,

$$\sup_{0 \le s \le t \le T} \frac{\left| B^{(H)}(t) - B^{(H)}(s) \right|}{\omega_H(t-s)} =: L < \infty,$$

where for $h > 0$

$$\omega_H(h) = h^H \sqrt{1 \vee \log h^{-1}}.$$

Thus the Landau and Shepp (1970) theorem implies that for appropriate constants $C > 0$ and $D > 0$, for all $t > 0$,

$$P\{L > t\} \le C \exp\left(-Dt^2\right).$$

A CLASS OF FUNCTIONS

For $K \geq 1$ denote the class of real-valued functions on $[0,T]$,

$$\mathcal{C}(K) = \{g : |g(s) - g(t)| \leq K\omega_H(|s-t|), 0 \leq s,t \leq T\}.$$

The class $\mathcal{C}(K)$ is closed in $\mathcal{C}[0,T]$. Consider the class of functions $\mathcal{C}[0,T] \to \mathbb{R}$ of the form

$$\mathcal{F}(K,\gamma) :=$$

$$\left\{h_{t,x}^{(K)}(g) = 1\{g(t) \leq x, g \in \mathcal{C}(K)\} : (t,x) \in \mathcal{T}(\gamma)\right\}.$$

It turns out that when $B_j^{(H)} \in \mathcal{C}(K)$, $j = 1, \ldots, n$

$$F_n(t,x) = n^{-1} \sum_{j=1}^{n} 1\left\{B_j^{(H)}(t) \leq x\right\}$$

$$= n^{-1} \sum_{j=1}^{n} h_{t,x}^{(K)}\left(B_j^{(H)}\right).$$

BRACKETING

Let $\mathcal{G}$ be a class of measurable real-valued functions defined on a measure space $(S, \mathcal{S})$. A way to measure the size of a class $\mathcal{G}$ is to use $L_2(P)$-brackets.

Let l and v be measurable real-valued functions on $(S, \mathcal{S})$ such that $l \leq v$ and

$$d_P(l, v) = \sqrt{\mathbb{E}_P \left(l\left(\xi\right) - v\left(\xi\right)\right)^2} < u, \; u > 0,$$

where ξ is a random variable taking values in S defined on a probability space $(\Omega, \mathcal{A}, P)$.

The pair of functions l, v form an u-bracket $[l, v]$ consisting of all the functions $f \in \mathcal{G}$ such that $l \leq f \leq v$.

Let $N_{[\,]}(u, \mathcal{G}, d_P)$ be the minimum number of u-brackets needed to cover $\mathcal{G}$.

FUNDAMENTAL TO OUR PROOF

Fundamental to our proof was a bound on

$$N_{[\,]}(u, \mathcal{F}(K, \gamma), d_P),$$

where P is the measure induced on the Borel sets of $\mathcal{C}[0, T]$, by $B^{(H)}$, with

$$d_P^2(l, v) = \mathbb{E}\left(l\left(B^{(H)}\right) - v\left(B^{(H)}\right)\right)^2.$$

This leads to needed moment bounds on the oscillations of our empirical and Gaussian processes.

BOUND ON TIES

LEMMA Let $B_j^{(H)}$, $j = 1, \ldots, n$, be i.i.d. fractional Brownian motions on $[0, \infty)$ with Hurst index $0 < H < 1$, where $n \geq 2\lceil 2/H \rceil + 2$, then with probability zero does there exist a subset $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$, where $m = 2\lceil 2/H \rceil + 2$ such that for some $t > 0$

$$B_{i_1}^{(H)}(t) = \cdots = B_{i_m}^{(H)}(t).$$

Proof If such a subset exists then the paths of the independent fractional Brownian motions in $\mathbb{R}^k$ with $2k = m$,

$$\left(B_{i_1}^{(H)}, \ldots, B_{i_k}^{(H)}\right) \text{ and } \left(B_{i_{k+1}}^{(H)}, \ldots, B_{i_{2k}}^{(H)}\right)$$

would have non-empty intersection except at 0, which, since $k > 2/H$, contradicts Theorem 3.2 in Xiao (1999). $\square$

XIAO (1999) RESULT

Theorem Let $X^1(t)$, $t \geq 0$, and $X^2(t)$, $t \geq 0$, be two independent fractional Brownian motions in R^d with index $0 < \alpha < 1$.

If $2/\alpha \leq d$, then with probability 1,

$$X^1([0,\infty)) \cap X^2((0,\infty)) = \varnothing.$$

We apply this result with

$$X^1 = \left(B_{i_1}^{(H)}, \ldots, B_{i_k}^{(H)}\right) \text{ and } X^2 = \left(B_{i_{k+1}}^{(H)}, \ldots, B_{i_{2k}}^{(H)}\right).$$