Precise moment and tail estimates for Rademacher sums in terms of weak parameters

Krzysztof Oleszkiewicz

University of Warsaw

High Dimensional Probability VII Conference
Cargése, May 26–31, 2014

Notation

Let $(F, \|\cdot\|)$ be a separable Banach space and let ξ be an F-valued random vector. For $p > 0$ let $\|\xi\|_p = (\mathbb{E}\|\xi\|^p)^{1/p}$ and let $\sigma_p(\xi) = \sup_{x^* \in F^*: \|x^*\| \leq 1} (\mathbb{E}|x^*(\xi)|^p)^{1/p}$. Obviously, $\|\xi\|_p \geq \sigma_p(\xi)$.

Let $v_1, v_2, \ldots, v_n \in F$. Let $r_1, r_2, \ldots, r_n, r'_1, r'_2, \ldots, r'_n, \varepsilon_1, \ldots, \varepsilon_n, \eta_1, \ldots, \eta_n, (X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be independent random variables. Distribution of $\mathbb{R}^2$-valued (X_i, Y_i)'s will be described later, the remaining random variables above are symmetric ± 1.

$g \sim \mathcal{N}(0, 1)$
$x_+ = \max(0, x)$.

Notation

Let $(F, \|\cdot\|)$ be a separable Banach space and let ξ be an F-valued random vector. For $p > 0$ let $\|\xi\|_p = (\mathbb{E}\|\xi\|^p)^{1/p}$ and let $\sigma_p(\xi) = \sup_{x^* \in F^* : \|x^*\| \leq 1} (\mathbb{E}|x^*(\xi)|^p)^{1/p}$. Obviously, $\|\xi\|_p \geq \sigma_p(\xi)$.

Let $v_1, v_2, \ldots, v_n \in F$. Let $r_1,\ r_2, \ldots, r_n,\ r'_1,\ r'_2, \ldots, r'_n,\ \varepsilon_1, \ldots, \varepsilon_n$, $\eta_1, \ldots, \eta_n$, $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be independent random variables. Distribution of $\mathbb{R}^2$-valued (X_i, Y_i)'s will be described later, the remaining random variables above are symmetric ± 1.

$g \sim \mathcal{N}(0, 1)$
$x_+ = \max(0, x)$.

Notation

Let $(F, \|\cdot\|)$ be a separable Banach space and let ξ be an F-valued random vector. For $p > 0$ let $\|\xi\|_p = (\mathbb{E}\|\xi\|^p)^{1/p}$ and let $\sigma_p(\xi) = \sup_{x^* \in F^* : \|x^*\| \le 1} (\mathbb{E}|x^*(\xi)|^p)^{1/p}$. Obviously, $\|\xi\|_p \ge \sigma_p(\xi)$.

Let $v_1, v_2, \ldots, v_n \in F$. Let $r_1,\, r_2, \ldots, r_n,\, r'_1,\, r'_2, \ldots, r'_n,\, \varepsilon_1, \ldots, \varepsilon_n$, $\eta_1, \ldots, \eta_n$, $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be independent random variables. Distribution of $\mathbb{R}^2$-valued (X_i, Y_i)'s will be described later, the remaining random variables above are symmetric ± 1.

$g \sim \mathcal{N}(0,1)$
$x_+ = \max(0, x)$.

Khinchine's inequality

$$S = \sum_{i=1}^{n} r_i v_i, \;\; S' = \sum_{i=1}^{n} r_i' v_i.$$

Khinchine's inequality: For $p > q > 0$ there exists some $C < \infty$ such that for any $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ we have

$$\|\sum_{i=1}^{n} a_i r_i\|_p \leq C \cdot \|\sum_{i=1}^{n} a_i r_i\|_q.$$

We denote by $K_{p,q}^{\mathbb{R}}$ the least (optimal) C for which Khinchine's inequality holds. Obviously, $\sigma_p(S) \leq K_{p,q}^{\mathbb{R}} \cdot \sigma_q(S)$ for S with vector coefficients.

We have $K_{p,q}^{\mathbb{R}} \geq \|g\|_p / \|g\|_q$ (by a variant of the CLT).

Khinchine's inequality

$$S = \sum_{i=1}^{n} r_i v_i, \;\; S' = \sum_{i=1}^{n} r_i' v_i.$$

Khinchine's inequality: For $p > q > 0$ there exists some $C < \infty$ such that for any $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ we have

$$\| \sum_{i=1}^{n} a_i r_i \|_p \leq C \cdot \| \sum_{i=1}^{n} a_i r_i \|_q.$$

We denote by $K^{\mathbb{R}}_{p,q}$ the least (optimal) C for which Khinchine's inequality holds. Obviously, $\sigma_p(S) \leq K^{\mathbb{R}}_{p,q} \cdot \sigma_q(S)$ for S with vector coefficients.

We have $K^{\mathbb{R}}_{p,q} \geq \|g\|_p / \|g\|_q$ (by a variant of the CLT).

Khinchine's inequality

$$S = \sum_{i=1}^{n} r_i v_i, \;\; S' = \sum_{i=1}^{n} r_i' v_i.$$

Khinchine's inequality: For $p > q > 0$ there exists some $C < \infty$ such that for any $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ we have

$$\| \sum_{i=1}^{n} a_i r_i \|_p \leq C \cdot \| \sum_{i=1}^{n} a_i r_i \|_q.$$

We denote by $K_{p,q}^{\mathbb{R}}$ the least (optimal) C for which Khinchine's inequality holds. Obviously, $\sigma_p(S) \leq K_{p,q}^{\mathbb{R}} \cdot \sigma_q(S)$ for S with vector coefficients.

We have $K_{p,q}^{\mathbb{R}} \geq \|g\|_p / \|g\|_q$ (by a variant of the CLT).

Khinchine's inequality

$$S = \sum_{i=1}^{n} r_i v_i, \;\; S' = \sum_{i=1}^{n} r_i' v_i.$$

Khinchine's inequality: For $p > q > 0$ there exists some $C < \infty$ such that for any $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ we have

$$\| \sum_{i=1}^{n} a_i r_i \|_p \leq C \cdot \| \sum_{i=1}^{n} a_i r_i \|_q.$$

We denote by $K^{\mathbb{R}}_{p,q}$ the least (optimal) C for which Khinchine's inequality holds. Obviously, $\sigma_p(S) \leq K^{\mathbb{R}}_{p,q} \cdot \sigma_q(S)$ for S with vector coefficients.

We have $K^{\mathbb{R}}_{p,q} \geq \|g\|_p / \|g\|_q$ (by a variant of the CLT).

Kahane's inequality:

For $p > q > 0$ there exists some $C < \infty$ such that for any $n \in \mathbb{N}$, any F and $v_1, v_2, \ldots, v_n \in F$ we have

$$\|\sum_{i=1}^{n} r_i v_i\|_p \leq C \cdot \|\sum_{i=1}^{n} r_i v_i\|_q.$$

We denote by $K_{p,q}$ the least (optimal) C for which Kahane's inequality holds. Obviously, $K^{\mathbb{R}}_{p,q} \leq K_{p,q}$.

Conjecture (Kwapień)

For every $p > q > 0$ there is $K^{\mathbb{R}}_{p,q} = K_{p,q}$.

Hypercontractivity: $K_{p,q} \leq \sqrt{\frac{p-1}{q-1}}$ for $p > q > 1$. Hence $K_{p,q}/K^{\mathbb{R}}_{p,q} \to 1$ as $p, q \to \infty$ since both $\sqrt{q-1}/\|g\|_q$ and $\sqrt{p-1}/\|g\|_p$ tend to $\mathrm{e}^{1/2}$.

Kahane's inequality:

For $p > q > 0$ there exists some $C < \infty$ such that for any $n \in \mathbb{N}$, any F and $v_1, v_2, \ldots, v_n \in F$ we have

$$\|\sum_{i=1}^{n} r_i v_i\|_p \leq C \cdot \|\sum_{i=1}^{n} r_i v_i\|_q.$$

We denote by $K_{p,q}$ the least (optimal) C for which Kahane's inequality holds. Obviously, $K_{p,q}^{\mathbb{R}} \leq K_{p,q}$.

Conjecture (Kwapień)

For every $p > q > 0$ there is $K_{p,q}^{\mathbb{R}} = K_{p,q}$.

Hypercontractivity: $K_{p,q} \leq \sqrt{\frac{p-1}{q-1}}$ for $p > q > 1$. Hence $K_{p,q}/K_{p,q}^{\mathbb{R}} \to 1$ as $p, q \to \infty$ since both $\sqrt{q-1}/\|g\|_q$ and $\sqrt{p-1}/\|g\|_p$ tend to $\mathrm{e}^{1/2}$.

Kahane's inequality:

For $p > q > 0$ there exists some $C < \infty$ such that for any $n \in \mathbb{N}$, any F and $v_1, v_2, \ldots, v_n \in F$ we have

$$\|\sum_{i=1}^{n} r_i v_i\|_p \leq C \cdot \|\sum_{i=1}^{n} r_i v_i\|_q.$$

We denote by $K_{p,q}$ the least (optimal) C for which Kahane's inequality holds. Obviously, $K_{p,q}^{\mathbb{R}} \leq K_{p,q}$.

Conjecture (Kwapień)

For every $p > q > 0$ there is $K_{p,q}^{\mathbb{R}} = K_{p,q}$.

Hypercontractivity: $K_{p,q} \leq \sqrt{\frac{p-1}{q-1}}$ for $p > q > 1$. Hence $K_{p,q}/K_{p,q}^{\mathbb{R}} \to 1$ as $p, q \to \infty$ since both $\sqrt{q-1}/\|g\|_q$ and $\sqrt{p-1}/\|g\|_p$ tend to $\mathrm{e}^{1/2}$.

Deviation bound

Theorem

For any $\delta \in (0,1]$ *and* $p \geq 1$ *we have*

$$\left\| \left(\|S\| - \frac{(1+\delta)^2}{\delta} \mathbb{E}\|S\| \right)_+ \right\|_p \leq (1+\delta)\sigma_p(S).$$

Corollary 1

There is

$$\sup_{q \in [1,p]} K_{p,q}/K^{\mathbb{R}}_{p,q} \to 1 \text{ as } p \to \infty.$$

Corollary 2

If a Rademacher series $S = \sum_{i=1}^{\infty} r_i v_i$ *is convergent a.s. then*

$$\|S\|_p/\sigma_p(S) \to 1 \text{ as } p \to \infty.$$

Deviation bound

Theorem

For any $\delta \in (0,1]$ *and* $p \geq 1$ *we have*

$$\left\| \left(\|S\| - \frac{(1+\delta)^2}{\delta} \mathbb{E}\|S\| \right)_+ \right\|_p \leq (1+\delta)\sigma_p(S).$$

Corollary 1

There is

$$\sup_{q \in [1,p]} K_{p,q}/K^{\mathbb{R}}_{p,q} \to 1 \text{ as } p \to \infty.$$

Corollary 2

If a Rademacher series $S = \sum_{i=1}^{\infty} r_i v_i$ *is convergent a.s. then*

$$\|S\|_p/\sigma_p(S) \to 1 \text{ as } p \to \infty.$$

Deviation bound

Theorem

For any $\delta \in (0,1]$ *and* $p \geq 1$ *we have*

$$\left\| \left(\|S\| - \frac{(1+\delta)^2}{\delta} \mathbb{E}\|S\| \right)_+ \right\|_p \leq (1+\delta)\sigma_p(S).$$

Corollary 1

There is

$$\sup_{q \in [1,p]} K_{p,q}/K^{\mathbb{R}}_{p,q} \to 1 \text{ as } p \to \infty.$$

Corollary 2

If a Rademacher series $S = \sum_{i=1}^{\infty} r_i v_i$ *is convergent a.s. then*

$$\|S\|_p/\sigma_p(S) \to 1 \text{ as } p \to \infty.$$

Proof of Corollary 1 – beginning

Proof of Corollary 1: For $p \geq 1$ there is

$$\|S\|_p \leq \sigma_p(S) + 2(\mathbb{E}\|S\|)^{1/2}(\mathbb{E}\|S\| + \sigma_p(S))^{1/2} + 2\mathbb{E}\|S\|.$$

Indeed, by Theorem we have

$$\|S\|_p \leq \frac{(1+\delta)^2}{\delta}\mathbb{E}\|S\| + \left\|\left(\|S\| - \frac{(1+\delta)^2}{\delta}\mathbb{E}\|S\|\right)_+\right\|_p$$

$$\leq \frac{(1+\delta)^2}{\delta}\mathbb{E}\|S\| + (1+\delta)\sigma_p(S)$$

and it suffices to optimize by setting

$$\delta = (\mathbb{E}\|S\|)^{1/2}(\mathbb{E}\|S\| + \sigma_p(S))^{-1/2}.$$

Proof of Corollary 1: For $p \geq 1$ there is

$$\|S\|_p \leq \sigma_p(S) + 2(\mathbb{E}\|S\|)^{1/2}(\mathbb{E}\|S\| + \sigma_p(S))^{1/2} + 2\mathbb{E}\|S\|.$$

Indeed, by Theorem we have

$$\|S\|_p \leq \frac{(1+\delta)^2}{\delta}\mathbb{E}\|S\| + \left\|\left(\|S\| - \frac{(1+\delta)^2}{\delta}\mathbb{E}\|S\|\right)_+\right\|_p$$

$$\leq \frac{(1+\delta)^2}{\delta}\mathbb{E}\|S\| + (1+\delta)\sigma_p(S)$$

and it suffices to optimize by setting

$$\delta = (\mathbb{E}\|S\|)^{1/2}(\mathbb{E}\|S\| + \sigma_p(S))^{-1/2}.$$

Proof of Corollary 1 – the end

Hence $K_{p,q} \leq K^{\mathbb{R}}_{p,q} + 2(1 + K^{\mathbb{R}}_{p,q})^{1/2} + 2$. Indeed,

$$\|S\|_p \leq \sigma_p(S) + 2(\mathbb{E}\|S\|)^{1/2}(\mathbb{E}\|S\| + \sigma_p(S))^{1/2} + 2\mathbb{E}\|S\| \leq$$

$$K^{\mathbb{R}}_{p,q} \cdot \sigma_q(S) + 2\|S\|_q^{1/2}(\|S\|_q + K^{\mathbb{R}}_{p,q} \cdot \sigma_q(S))^{1/2} + 2\|S\|_q$$

$$\leq (K^{\mathbb{R}}_{p,q} + 2(1 + K^{\mathbb{R}}_{p,q})^{1/2} + 2) \cdot \|S\|_q.$$

For $q \in [1, p^{1/5}]$ we have $K^{\mathbb{R}}_{p,q} \geq \|g\|_p / \|g\|_q \to \infty$, which ends the proof, while for $q \in [p^{1/5}, p]$ both p and q tend to infinity, so that Corollary 1 follows by the hypercontractivity estimates.

Proof of Corollary 1 – the end

Hence $K_{p,q} \le K^{\mathbb{R}}_{p,q} + 2(1 + K^{\mathbb{R}}_{p,q})^{1/2} + 2$. Indeed,

$$\|S\|_p \le \sigma_p(S) + 2(\mathbb{E}\|S\|)^{1/2}(\mathbb{E}\|S\| + \sigma_p(S))^{1/2} + 2\mathbb{E}\|S\| \le$$

$$K^{\mathbb{R}}_{p,q} \cdot \sigma_q(S) + 2\|S\|_q^{1/2}(\|S\|_q + K^{\mathbb{R}}_{p,q} \cdot \sigma_q(S))^{1/2} + 2\|S\|_q$$

$$\le (K^{\mathbb{R}}_{p,q} + 2(1 + K^{\mathbb{R}}_{p,q})^{1/2} + 2) \cdot \|S\|_q.$$

For $q \in [1, p^{1/5}]$ we have $K^{\mathbb{R}}_{p,q} \ge \|g\|_p / \|g\|_q \to \infty$, which ends the proof, while for $q \in [p^{1/5}, p]$ both p and q tend to infinity, so that Corollary 1 follows by the hypercontractivity estimates.

Proof of Corollary 1 – the end

Hence $K_{p,q} \leq K^{\mathbb{R}}_{p,q} + 2(1 + K^{\mathbb{R}}_{p,q})^{1/2} + 2$. Indeed,

$$\|S\|_p \leq \sigma_p(S) + 2(\mathbb{E}\|S\|)^{1/2}(\mathbb{E}\|S\| + \sigma_p(S))^{1/2} + 2\mathbb{E}\|S\| \leq$$

$$K^{\mathbb{R}}_{p,q} \cdot \sigma_q(S) + 2\|S\|_q^{1/2}(\|S\|_q + K^{\mathbb{R}}_{p,q} \cdot \sigma_q(S))^{1/2} + 2\|S\|_q$$

$$\leq (K^{\mathbb{R}}_{p,q} + 2(1 + K^{\mathbb{R}}_{p,q})^{1/2} + 2) \cdot \|S\|_q.$$

For $q \in [1, p^{1/5}]$ we have $K^{\mathbb{R}}_{p,q} \geq \|g\|_p / \|g\|_q \to \infty$, which ends the proof, while for $q \in [p^{1/5}, p]$ both p and q tend to infinity, so that Corollary 1 follows by the hypercontractivity estimates.

Proof of Corollary 2: As we already know,

$$\|S\|_p \le \sigma_p(S) + 2(\mathbb{E}\|S\|)^{1/2}(\mathbb{E}\|S\| + \sigma_p(S))^{1/2} + 2\mathbb{E}\|S\|.$$

Thus $\sigma_p(S) \to \infty$ implies $\|S\|_p/\sigma_p(S) \to 1$ as $p \to \infty$. Now assume

$$\sigma_\infty(S) := \lim_{p\to\infty} \sigma_p(S) < \infty,$$

so that $\|S\|_\infty < \infty$ as well. Let $\delta > 0$. $\mathbb{P}(\|S\| > \|S\|_\infty - \delta) > 0$, so $\exists\ n \in \mathbb{N}$ such that $\mathbb{P}(\|S_n\| > \|S\|_\infty - 2\delta) > 0$. Hence $\exists_\omega\ \|S_n(\omega)\| > \|S\|_\infty - 2\delta$ and $\exists\ x^*_\omega$ with norm 1 and such that $x^*_\omega(S_n(\omega)) > \|S\|_\infty - \delta$, $\mathbb{P}(x^*_\omega(S) > \|S\|_\infty - \delta) > 0$, and therefore $\sigma_\infty(S) \ge \|S\|_\infty - \delta \to \|S\|_\infty$ as $\delta \to 0$.

Proof of Corollary 2: As we already know,

$$\|S\|_p \leq \sigma_p(S) + 2(\mathbb{E}\|S\|)^{1/2}(\mathbb{E}\|S\| + \sigma_p(S))^{1/2} + 2\mathbb{E}\|S\|.$$

Thus $\sigma_p(S) \to \infty$ implies $\|S\|_p/\sigma_p(S) \to 1$ as $p \to \infty$. Now assume

$$\sigma_\infty(S) := \lim_{p\to\infty} \sigma_p(S) < \infty,$$

so that $\|S\|_\infty < \infty$ as well. Let $\delta > 0$. $\mathbb{P}(\|S\| > \|S\|_\infty - \delta) > 0$, so $\exists\ n \in \mathbb{N}$ such that $\mathbb{P}(\|S_n\| > \|S\|_\infty - 2\delta) > 0$. Hence $\exists_\omega\ \|S_n(\omega)\| > \|S\|_\infty - 2\delta$ and $\exists\ x^*_\omega$ with norm 1 and such that $x^*_\omega(S_n(\omega)) > \|S\|_\infty - \delta$, $\mathbb{P}(x^*_\omega(S) > \|S\|_\infty - \delta) > 0$, and therefore $\sigma_\infty(S) \geq \|S\|_\infty - \delta \to \|S\|_\infty$ as $\delta \to 0$.

Proof of Corollary 2: As we already know,

$$\|S\|_p \leq \sigma_p(S) + 2(\mathbb{E}\|S\|)^{1/2}(\mathbb{E}\|S\| + \sigma_p(S))^{1/2} + 2\mathbb{E}\|S\|.$$

Thus $\sigma_p(S) \to \infty$ implies $\|S\|_p/\sigma_p(S) \to 1$ as $p \to \infty$. Now assume

$$\sigma_\infty(S) := \lim_{p\to\infty} \sigma_p(S) < \infty,$$

so that $\|S\|_\infty < \infty$ as well. Let $\delta > 0$. $\mathbb{P}(\|S\| > \|S\|_\infty - \delta) > 0$, so $\exists\ n \in \mathbb{N}$ such that $\mathbb{P}(\|S_n\| > \|S\|_\infty - 2\delta) > 0$. Hence $\exists_\omega\ \|S_n(\omega)\| > \|S\|_\infty - 2\delta$ and $\exists\ x^*_\omega$ with norm 1 and such that $x^*_\omega(S_n(\omega)) > \|S\|_\infty - \delta$, $\mathbb{P}(x^*_\omega(S) > \|S\|_\infty - \delta) > 0$, and therefore $\sigma_\infty(S) \geq \|S\|_\infty - \delta \to \|S\|_\infty$ as $\delta \to 0$.

Proof of Corollary 2: As we already know,

$$\|S\|_p \leq \sigma_p(S) + 2(\mathbb{E}\|S\|)^{1/2}(\mathbb{E}\|S\| + \sigma_p(S))^{1/2} + 2\mathbb{E}\|S\|.$$

Thus $\sigma_p(S) \to \infty$ implies $\|S\|_p/\sigma_p(S) \to 1$ as $p \to \infty$. Now assume

$$\sigma_\infty(S) := \lim_{p\to\infty} \sigma_p(S) < \infty,$$

so that $\|S\|_\infty < \infty$ as well. Let $\delta > 0$. $\mathbb{P}(\|S\| > \|S\|_\infty - \delta) > 0$, so $\exists\ n \in \mathbb{N}$ such that $\mathbb{P}(\|S_n\| > \|S\|_\infty - 2\delta) > 0$. Hence $\exists_\omega\ \|S_n(\omega)\| > \|S\|_\infty - 2\delta$ and $\exists\ x^*_\omega$ with norm 1 and such that $x^*_\omega(S_n(\omega)) > \|S\|_\infty - \delta$, $\mathbb{P}(x^*_\omega(S) > \|S\|_\infty - \delta) > 0$, and therefore $\sigma_\infty(S) \geq \|S\|_\infty - \delta \to \|S\|_\infty$ as $\delta \to 0$.

Contraction principle

Lemma 1: Let $c > 0$. For any convex function $\Phi : [-c, c] \longrightarrow \mathbb{R}$ and any mean zero real random variable Y such that $|Y| \le c$ a.s. we have

$$\mathbb{E}\Phi(Y) \le \frac{\Phi(c) + \Phi(-c)}{2}.$$

Proof of Lemma 1:

$$\mathbb{E}\Phi(Y) = \mathbb{E}\Phi\Big(\frac{c+Y}{2c} \cdot c + \frac{c-Y}{2c} \cdot (-c)\Big) \le$$

$$\mathbb{E}\Big(\frac{c+Y}{2c}\Phi(c) + \frac{c-Y}{2c}\Phi(-c)\Big) = \frac{\Phi(c)+\Phi(-c)}{2}.$$

Lemma 1: Let $c > 0$. For any convex function $\Phi : [-c, c] \longrightarrow \mathbb{R}$ and any mean zero real random variable Y such that $|Y| \leq c$ a.s. we have

$$\mathbb{E}\Phi(Y) \leq \frac{\Phi(c) + \Phi(-c)}{2}.$$

Proof of Lemma 1:

$$\mathbb{E}\Phi(Y) = \mathbb{E}\Phi\Big(\frac{c+Y}{2c} \cdot c + \frac{c-Y}{2c} \cdot (-c)\Big) \leq$$

$$\mathbb{E}\Big(\frac{c+Y}{2c}\Phi(c) + \frac{c-Y}{2c}\Phi(-c)\Big) = \frac{\Phi(c) + \Phi(-c)}{2}.$$

Lemma 2: Let $\Phi : [-c, c]^n \longrightarrow \mathbb{R}$ be convex and assume that $Y_1, Y_2, \ldots, Y_n$ are independent random variables with $\mathbb{E} Y_i = 0$ and $|Y_i| \leq c$ a.s. for $i \leq n$. Then $\mathbb{E}\Phi(Y_1, \ldots, Y_n) \leq \mathbb{E}\Phi(c\eta_1, \ldots, c\eta_n)$.

Proof of Lemma 2: It follows from Lemma 1 by an obvious induction. In fact it suffices to assume Φ is convex with respect to every coordinate.

Lemma 2: Let $\Phi : [-c, c]^n \longrightarrow \mathbb{R}$ be convex and assume that $Y_1, Y_2, \ldots, Y_n$ are independent random variables with $\mathbb{E} Y_i = 0$ and $|Y_i| \leq c$ a.s. for $i \leq n$. Then $\mathbb{E}\Phi(Y_1, \ldots, Y_n) \leq \mathbb{E}\Phi(c\eta_1, \ldots, c\eta_n)$.

Proof of Lemma 2: It follows from Lemma 1 by an obvious induction. In fact it suffices to assume Φ is convex with respect to every coordinate.

Random vectors (X_i, Y_i)

Let distribution of $(X_1, Y_1), \ldots, (X_n, Y_n)$ be given by

$$\mathbb{P}\Big((X_i, Y_i) = (1, 1-\delta)\Big) = \frac{1}{2},$$

$$\mathbb{P}\Big((X_i, Y_i) = (1, 1+\delta)\Big) = \frac{\delta}{2(1+\delta)},$$

$$\mathbb{P}\Big((X_i, Y_i) = (-1, -1-\delta)\Big) = \frac{1}{2(1+\delta)}.$$

It is easy to check that $\mathbb{E}X_i = \frac{\delta}{1+\delta}$, $\mathbb{E}Y_i = 0$ and $|Y_i| \leq 1 + \delta$ a.s.
It is quite obvious that $(\varepsilon_i X_i, \varepsilon_i Y_i) \sim (r_i, r_i - \delta r_i')$, so that

$$(S, S - \delta S') \sim (\sum_{i=1}^{n} \varepsilon_i X_i v_i, \sum_{i=1}^{n} \varepsilon_i Y_i v_i).$$

Random vectors (X_i, Y_i)

Let distribution of $(X_1, Y_1), \ldots, (X_n, Y_n)$ be given by

$$\mathbb{P}\Big((X_i, Y_i) = (1, 1-\delta)\Big) = \frac{1}{2},$$

$$\mathbb{P}\Big((X_i, Y_i) = (1, 1+\delta)\Big) = \frac{\delta}{2(1+\delta)},$$

$$\mathbb{P}\Big((X_i, Y_i) = (-1, -1-\delta)\Big) = \frac{1}{2(1+\delta)}.$$

It is easy to check that $\mathbb{E}X_i = \frac{\delta}{1+\delta}$, $\mathbb{E}Y_i = 0$ and $|Y_i| \leq 1+\delta$ a.s. It is quite obvious that $(\varepsilon_i X_i, \varepsilon_i Y_i) \sim (r_i, r_i - \delta r_i')$, so that

$$(S, S - \delta S') \sim (\sum_{i=1}^{n} \varepsilon_i X_i v_i, \sum_{i=1}^{n} \varepsilon_i Y_i v_i).$$

Proof of deviation bound – Hahn-Banach starter

Proof of Theorem: Let $x_v^* \in F^*$ be such that $\|x_v^*\| = 1$ and $x_v^*(v) = \|v\|$. For every $v \in F$ we have

$$\|v\| - \|v - \delta S\| = x_v^*(v) - \|v - \delta S\| \leq x_v^*(v) - x_v^*(v - \delta S) = \delta \cdot x_v^*(S).$$

Hence for every $v \in F$ there is

$$\mathbb{E}(\|v\| - \|v - \delta S\|)_+^p \leq \delta^p \cdot \mathbb{E}|x_v^*(S)|^p \leq \delta^p \cdot \sigma_p(S)^p,$$

so that

$$\mathbb{E}(\|S\| - \|S - \delta S'\|)_+^p \leq \delta^p \cdot \sigma_p(S)^p.$$

Proof of deviation bound – Hahn-Banach starter

Proof of Theorem: Let $x_v^* \in F^*$ be such that $\|x_v^*\| = 1$ and $x_v^*(v) = \|v\|$. For every $v \in F$ we have

$$\|v\| - \|v - \delta S\| = x_v^*(v) - \|v - \delta S\| \leq x_v^*(v) - x_v^*(v - \delta S) = \delta \cdot x_v^*(S).$$

Hence for every $v \in F$ there is

$$\mathbb{E}(\|v\| - \|v - \delta S\|)_+^p \leq \delta^p \cdot \mathbb{E}|x_v^*(S)|^p \leq \delta^p \cdot \sigma_p(S)^p,$$

so that

$$\mathbb{E}(\|S\| - \|S - \delta S'\|)_+^p \leq \delta^p \cdot \sigma_p(S)^p.$$

Proof of deviation bound – Hahn-Banach starter

Proof of Theorem: Let $x_v^* \in F^*$ be such that $\|x_v^*\| = 1$ and $x_v^*(v) = \|v\|$. For every $v \in F$ we have

$$\|v\| - \|v - \delta S\| = x_v^*(v) - \|v - \delta S\| \leq x_v^*(v) - x_v^*(v - \delta S) = \delta \cdot x_v^*(S).$$

Hence for every $v \in F$ there is

$$\mathbb{E}(\|v\| - \|v - \delta S\|)_+^p \leq \delta^p \cdot \mathbb{E}|x_v^*(S)|^p \leq \delta^p \cdot \sigma_p(S)^p,$$

so that

$$\mathbb{E}(\|S\| - \|S - \delta S'\|)_+^p \leq \delta^p \cdot \sigma_p(S)^p.$$

Proof of deviation bound – continued

Therefore

$$\mathbb{E}(\|\sum_{i=1}^{n}\varepsilon_i X_i v_i\| - \|\sum_{i=1}^{n}\varepsilon_i Y_i v_i\|)_+^p \leq \delta^p \cdot \sigma_p(S)^p.$$

Since $x \mapsto x_+^p$ is nondecreasing, convex, by Jensen's inequality we get

$$\mathbb{E}(\|\sum_{i=1}^{n}\varepsilon_i X_i v_i\| - \|\sum_{i=1}^{n}\varepsilon_i Y_i v_i\|)_+^p \geq$$

$$\mathbb{E}\left(\mathbb{E}_{X,Y}\|\sum_{i=1}^{n}\varepsilon_i X_i v_i\| - \mathbb{E}_{X,Y}\|\sum_{i=1}^{n}\varepsilon_i Y_i v_i\|\right)_+^p,$$

where $\mathbb{E}_{X,Y} = \mathbb{E}(\cdot \,|\, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ – recall that (X_i, Y_i)'s are independent of $\sigma(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$.

Proof of deviation bound – continued

Therefore

$$\mathbb{E}(\|\sum_{i=1}^{n}\varepsilon_i X_i v_i\| - \|\sum_{i=1}^{n}\varepsilon_i Y_i v_i\|)_+^p \le \delta^p \cdot \sigma_p(S)^p.$$

Since $x \mapsto x_+^p$ is nondecreasing, convex, by Jensen's inequality we get

$$\mathbb{E}(\|\sum_{i=1}^{n}\varepsilon_i X_i v_i\| - \|\sum_{i=1}^{n}\varepsilon_i Y_i v_i\|)_+^p \ge$$

$$\mathbb{E}\Big(\mathbb{E}_{X,Y}\|\sum_{i=1}^{n}\varepsilon_i X_i v_i\| - \mathbb{E}_{X,Y}\|\sum_{i=1}^{n}\varepsilon_i Y_i v_i\|\Big)_+^p,$$

where $\mathbb{E}_{X,Y} = \mathbb{E}(\,\cdot \,|\, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ – recall that (X_i, Y_i)'s are independent of $\sigma(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$.

Proof of deviation bound – convexity

By the convexity of the norm we get

$$\mathbb{E}_{X,Y}\|\sum_{i=1}^{n}\varepsilon_i X_i v_i\| \geq \|\sum_{i=1}^{n}\varepsilon_i \cdot \mathbb{E}X_i \cdot v_i\| = \frac{\delta}{1+\delta}\|\sum_{i=1}^{n}\varepsilon_i v_i\|.$$

On the other hand, by using Lemma 2 for $\Phi(y_1,\ldots,y_n) = \|\sum_{i=1}^{n}\varepsilon_i y_i v_i\|$, we get

$$\mathbb{E}_{X,Y}\|\sum_{i=1}^{n}\varepsilon_i Y_i v_i\| \leq \mathbb{E}_{\eta}\|\sum_{i=1}^{n}\varepsilon_i(1+\delta)\eta_i v_i\| = (1+\delta)\mathbb{E}\|S\|$$

since $(\varepsilon_1\cdot\eta_1,\ldots,\varepsilon_n\cdot\eta_n) \sim (\eta_1,\ldots,\eta_n)$ for any fixed sequence of signs ε_i's.

Proof of deviation bound – convexity

By the convexity of the norm we get

$$\mathbb{E}_{X,Y}\|\sum_{i=1}^{n}\varepsilon_i X_i v_i\| \geq \|\sum_{i=1}^{n}\varepsilon_i \cdot \mathbb{E}X_i \cdot v_i\| = \frac{\delta}{1+\delta}\|\sum_{i=1}^{n}\varepsilon_i v_i\|.$$

On the other hand, by using Lemma 2 for $\Phi(y_1,\ldots,y_n) = \|\sum_{i=1}^{n}\varepsilon_i y_i v_i\|$, we get

$$\mathbb{E}_{X,Y}\|\sum_{i=1}^{n}\varepsilon_i Y_i v_i\| \leq \mathbb{E}_\eta\|\sum_{i=1}^{n}\varepsilon_i(1+\delta)\eta_i v_i\| = (1+\delta)\mathbb{E}\|S\|$$

since $(\varepsilon_1\cdot\eta_1,\ldots,\varepsilon_n\cdot\eta_n) \sim (\eta_1,\ldots,\eta_n)$ for any fixed sequence of signs ε_i's.

Proof of deviation bound – the end

Hence

$$\mathbb{E}\Big(\frac{\delta}{1+\delta}\,\|\sum_{i=1}^{n}\varepsilon_i v_i\| - (1+\delta)\mathbb{E}\|S\|\Big)_+^p \le \delta^p \cdot \sigma_p(S)^p$$

and therefore

$$\Big\|\Big(\|S\| - \frac{(1+\delta)^2}{\delta}\mathbb{E}\|S\|\Big)_+\Big\|_p \le (1+\delta)\sigma_p(S).$$

The proof is finished.

Proof of deviation bound – the end

Hence

$$\mathbb{E}\Big(\frac{\delta}{1+\delta}\,\|\sum_{i=1}^{n}\varepsilon_i v_i\| - (1+\delta)\mathbb{E}\|S\|\Big)_+^p \le \delta^p \cdot \sigma_p(S)^p$$

and therefore

$$\Big\|\Big(\|S\| - \frac{(1+\delta)^2}{\delta}\mathbb{E}\|S\|\Big)_+\Big\|_p \le (1+\delta)\sigma_p(S).$$

The proof is finished.

The complex case:
In the complex Banach space one can replace ± 1 random variables by Steinhaus random variables distributed uniformly over the unit circle $\{z : |z| = 1\}$. Theorem and both Corollaries can be transferred to the complex setting, up to some minor changes (Lemma 2 is the only place which needs a slightly different approach).

Tail estimates:
The proof technique may be modified to yield tight tail (instead of: moment) estimates.

Extensions

The complex case:
In the complex Banach space one can replace ± 1 random variables by Steinhaus random variables distributed uniformly over the unit circle $\{z : |z| = 1\}$. Theorem and both Corollaries can be transferred to the complex setting, up to some minor changes (Lemma 2 is the only place which needs a slightly different approach).

Tail estimates:
The proof technique may be modified to yield tight tail (instead of: moment) estimates.