Refinements of the Free Poincare Inequality

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Cargèse, May 26-30, 2014

Classical Case

A probability measure μ on $\mathbb{R}^d$ satisfies the Poincaré's inequality if there exists $\rho > 0$ such that

$$2\rho \mathrm{Var}_\mu(f) \leq \int |\nabla f|^2 d\mu, \text{ for all nice } f. \qquad (P(\rho))$$

$$\mathrm{Var}_\mu(f) = \int \left(f - \int f d\mu\right)^2 d\mu = \int f^2 d\mu - \left(\int f d\mu\right)^2.$$

Reinterpretations:

1.
$$2\rho\Pi \leq L \qquad (P(\rho))$$

$$\mathrm{Var}_\mu(f) = \langle \Pi f, f\rangle_{L^2(\mu)} \quad \Pi f = f - \int f d\mu$$

   with Π the projection onto the orthogonal to constants and

$$\langle Lf, f\rangle_{L^2(\mu)} = \int |\nabla f|^2 d\mu \geq 0.$$

2. $spec(L) \subset \{0\} \cup [2\rho, \infty)$, i.e. spectral gap of size 2ρ for L.

Example: The Gaussian Case

$\gamma(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$

Hermite functions: $\phi_n(x) = \frac{(-1)^n}{\sqrt{n!}} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \quad n \geq 0$ form a basis of $L^2(\gamma)$.

$$\int |f'|^2 d\gamma = \langle Lf, f \rangle_{L^2(\gamma)}, \quad Lf(x) = -f''(x) + xf'(x), \quad L\phi_n = n\phi_n.$$

and for $f = \sum_{n \geq 0} \alpha_n \phi_n$,

$$\mathrm{Var}_\gamma(f) \leq \int |f'|^2 d\gamma$$

$$\langle \Pi f, f \rangle_{L^2(\gamma)} \leq \langle Lf, f \rangle_{L^2(\gamma)}$$

$$\sum_{n \geq 1} \alpha_n^2 \leq \sum_{n \geq 1} n\alpha_n^2.$$

Hence

$$\mathrm{Var}_\gamma(f) \leq \int (f')^2 d\gamma. \qquad (P(1/2))$$

Brascamp-Lieb Inequality

Theorem

If $\mu(dx) = e^{-V(x)}dx$ on $\mathbb{R}^d$ with $HessV(x) > 0$ for any x, then

$$\mathrm{Var}_\mu(f) \leq \int \langle (HessV)^{-1}\nabla f, \nabla f\rangle d\mu.$$

In particular, if $V(x) - \rho|x|^2$ is convex we recover $P(\rho)$ for μ.

The idea (due to Helffer). Take $L = -\Delta + \nabla V \cdot \nabla$. $L \geq 0$ and

$$Var_\mu(f) = \langle (L + V'')^{-1}\nabla f, \nabla f\rangle_{L^2(\mu)} \leq \langle (V'')^{-1}\nabla f, \nabla f\rangle_{L^2(\mu)}.$$

Houdré-Kagan inequality

Theorem

If γ is the standard normal dist on $\mathbb{R}$, then

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k!} \int |\phi^{(k)}|^2 d\gamma \leq \mathrm{Var}_\gamma(\phi) \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} \int |\phi^{(k)}|^2 d\gamma.$$

for any $n \geq 1$.

For $n = 1$,

$$\int |\phi'|^2 d\gamma - \frac{1}{2} \int \|\phi''\|^2 d\gamma \leq \mathrm{Var}_\gamma(\phi) \leq \int |\phi'|^2 d\gamma.$$

The original proof uses the Hermite polynomials expansion.
M.Ledoux has a purely semigroup proof.

A little extension of Brascamp-Lieb in the spirit of Houdre-Kagan

Theorem (IP(2013))

On the real line, if $d\mu = e^{-V}dx$ is a probability, then

$$Var_\mu(f) = \|E_1^{-1/2} f'\|^2 - \|E_2^{-1/2} f_2\|^2 + \cdots + (-1)^{n-1}\|E_n^{-1/2} f_n\|^2 + (-1)^n \langle (I + DE_n^{-1}D^*)^{-1} D[E_n^{-1} f_n], D[E_n^{-1} f_n]\rangle.$$

with

$$E_1 = V'' \text{ and } f_1 = E_1^{-1} f'$$
$$E_n = E_{n-1}(1 + \mathcal{A}(E_{n-1}^{-1})) \text{ and } f_n = E_{n-1}^{1/2} D[f_{n-1}]$$
$$\mathcal{A}(E)(x) = \frac{1}{4}\left(2E''(x) + 2V'(x)E'(x) - \frac{E'(x)^2}{E(x)} + 4E(x)V''(x)\right).$$

The idea:
$Var_\mu(f) = \langle (L + V'')^{-1} f', f'\rangle_{L^2(\mu)} = \langle (V'')^{-1} f', f'\rangle - \langle [(V'')^{-1} - (L + V'')^{-1}] f', f'\rangle_{L^2(\mu)} \ldots$

A first version of a free Poincaré

Theorem (Biane (2003))

If $\alpha(dx) = \mathbb{1}_{[-2,2]}(x)\frac{\sqrt{4-x^2}dx}{2\pi}$, *then*

$$\mathrm{Var}_\alpha(f) \leq \iint \left(\frac{f(x)-f(y)}{x-y}\right)^2 \alpha(dx)\alpha(dy).$$

Essentially the same as the classical one with the derivative replaced by the non-commutative derivative.

The proof: The operator $\mathcal{M}$, whose Dirichlet form is given by the right hand side is the counting number operator for the Chebyshev polynomials of the second kind.
$\mathcal{M} = \partial^*\partial$ is the Ornestein-Uhlenbeck operator.

A second version of free Poincaré

Theorem (M. Ledoux and I.P. (2009))

$$\int_{-2}^{2}\int_{-2}^{2}\left(\frac{f(x)-f(y)}{x-y}\right)^2 \frac{(4-xy)dxdy}{4\pi^2\sqrt{(4-x^2)(4-y^2)}} \le \int (f')^2\, d\alpha. \qquad (*)$$

First proof: Poincaré+Fluctuations for random matrices.

Second Proof: If $\mathcal{N}$ is the counting number operator for the Chebyshev polynomials of the first kind

$$(\mathcal{N}\phi)(x) = \int (x+y)\phi'(y)\beta(dy) - (4-x^2)\int \frac{\phi'(x)-\phi'(y)}{x-y}\beta(dy).$$

then (*) is the same as

$$\langle \mathcal{N}f, f\rangle_{L^2(\beta)} \le \langle \mathcal{L}f, f\rangle_{L^2(\beta)}, \quad \text{with} \quad \beta(dx) = \mathbb{1}_{[-2,2]}(dx)\frac{dx}{\pi\sqrt{4-x^2}},$$

and $\mathcal{L}f = \mathcal{N}^2 f = -(4-x^2)f''(x) + xf'(x)$ a Jacobi type operator.

Which version is the true one?

- Poincaré in large dimensions imply the second version of free Poincaré.
- Free transportation and Log-Sobolev imply the second version of free Poincaré.

Weighted Logarithmic Potentials (Free Entropy)

V smooth such that $\lim_{|x|\to\infty} \frac{V(x)}{\log(1+|x|^2)} = \infty$.

$$E_V(\mu) = \int V(x)\mu(x) - \iint \log|x-y|\mu(dx)\mu(dy).$$

There is a unique probability measure μ_V such that

$$E_V(\mu_V) = \inf_{\mu\in\mathcal{P}(\mathbb{R})} E_V(\mu).$$

In addition, μ_V has compact support.
If $V(x) = x^2/2$, $\mu_V(dx) = \mathbb{1}_{[-2,2]}\frac{\sqrt{4-x^2}}{2\pi}dx$.
The relative free entropy is defined as

$$E_V(\mu|\mu_V) = E_V(\mu) - E_V(\mu_V).$$

It is always positive, unless $\mu = \mu_V$.

One Dimensional Free Information

$$I_V(\mu|\mu_V) = \int (H\mu(x) - V'(x))^2 \mu(dx)$$

where $H\mu(x) = \int \frac{2}{x-y} \mu(dx)$ in the principal value sense.
In the case $V(x) = x^2/2$,

$$I(\mu|\mu_V) = \int (H\mu(x) - x)^2 \mu(dx).$$

Free LSI for the potential V states that there exists $\rho > 0$ such that

$$E_V(\mu|\mu_V) \leq \frac{1}{4\rho} I_V(\mu|\mu_V).$$

for any probability μ.

The One Dimensional Free Poincaré

We say that the probability measure μ supported on $[-2,2]$ satisfies $P(\rho)$, $\rho > 0$, if

$$2\rho \iint \left(\frac{f(x) - f(y)}{x - y} \right)^2 \frac{(4 - xy)dxdy}{4\pi^2 \sqrt{(4 - x^2)(4 - y^2)}} \leq \int (f')^2 \, d\mu \qquad (P(\rho))$$

If μ has $P(\rho)$, then necessarily its support is the whole $[-2,2]$. In fact, the density of μ w.r.t. $\alpha(dx) = \mathbb{1}_{[-2,2]} \frac{\sqrt{4-x^2}}{2\pi} dx$ must be positive on $[-2,2]$.

Theorem (M.Ledoux and I.P. (2011))

If μ_V is the equilibrium measure of E_V for a C^3 potential V such that $\frac{d\mu_V}{d\alpha} > 0$ on $[-2,2]$, then the free LSI(ρ) implies the free $P(\rho)$ for μ_V.

The free Brascamp-Lieb

Theorem (C. Houdre and I.P.)

If μ_V is the equilibrium measure of a convex potential V supported on $[-2,2]$, then

$$\int_{-2}^{2}\int_{-2}^{2}\left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2 \frac{4-xy}{4\pi^2\sqrt{(4-x^2)(4-y^2)}}dxdy \le \int \frac{\phi'^2}{V''}d\mu_V$$

for any smooth function ϕ on $[-2,2]$.

The idea of the proof is that the left hand side equals

$$\langle (\mathcal{M}_V + V'')^{-1}\phi', \phi'\rangle_{\mu_V}$$

where $\mathcal{M}_V = \partial^*\partial$ where ∂ is the noncommutative derivative introduced by Voiculescu and the ∂^* is the adjoint with respect to $\mu_V \otimes \mu_V$. Since $\mathcal{M}_V \ge 0$, this implies the inequality.

An extension for the semicircular measure

Theorem (C. Houdre and I.P.)

For any smooth function ϕ on $[-2, 2]$, and any positive integer k,

$$\sum_{l=1}^{2k} \frac{(-1)^{l-1}}{l} \|\partial^{(l-1)}\phi'\|^2_{\alpha^{\otimes l}} \leq \frac{1}{2}\langle \mathcal{N}\phi, \phi\rangle \leq \sum_{l=1}^{2k-1} \frac{(-1)^{l-1}}{l} \|\partial^{(l-1)}\phi'\|^2_{\alpha^{\otimes l}},$$

where $\partial^{(l)}$ are the non-commutative derivatives introduced by Voiculescu and

$$\frac{1}{2}\langle \mathcal{N}\phi, \phi\rangle = \int_{-2}^{2}\int_{-2}^{2} \left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2 \frac{(4-xy)dxdy}{4\pi^2\sqrt{(4-x^2)(4-y^2)}}$$

The idea of the proof for $k = 2$ is based on the fact that

$$\begin{aligned}\langle \mathcal{N}\phi, \phi\rangle/2 &= \langle (\mathcal{M}+I)^{-1}\phi', \phi'\rangle_\alpha = \langle \phi', \phi'\rangle - \langle \mathcal{M}(\mathcal{M}+I)^{-1}\phi', \phi'\rangle_\alpha \\ &= \langle \phi', \phi'\rangle - \frac{1}{2}\langle \partial\phi', \partial\phi'\rangle_{\alpha^{\otimes 2}} + \frac{1}{2}\langle \mathcal{M}^{(2)}(\mathcal{M}^{(2)}+2I)^{-1}\partial\phi', \partial\phi'\rangle_{\alpha^{\otimes 2}}\end{aligned}$$

with $\mathcal{M}^2(\phi \otimes \psi) = (\mathcal{M}\phi) \otimes \psi + \phi \otimes (\mathcal{M}\psi)$.

A Poincaré inquality for Marcenko-Pastur

Theorem (M.Ledoux and I.P. (2009))

Assume that $Q : [0, \infty) \to \mathbb{R}$ is a convex potential and let $V(x) = Q(x) - s\log(x)$ for $s > 0$ satisfy $\lim_{x\to\infty}(V(x) - 2\log(x)) = \infty$. Assume that the support of μ_V is $[a, b]$. Then for any smooth function ϕ on $[a, b]$:

$$\int x^2\phi'(x)^2\mu_V(dx) \geq \frac{s}{4\pi^2}\int_a^b\int_a^b\left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2\frac{-2ab+(a+b)(x+y)-2xy}{2\sqrt{(x-a)(b-x)}\sqrt{(y-a)(b-y)}}dxdy.$$

If $Q(x) = rx + t$, equality is attained for $\phi(x) = c_1 + \frac{c_2}{x}$.
In the case $V(x) = rx$, $r > 0$, on $[0, \infty)$, for every smooth ϕ on $[a, b]$, the following holds,

$$\int x\phi'(x)^2\mu_V(dx) \geq \frac{r}{4\pi^2}\int_a^b\int_a^b\left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2\frac{-2ab+(a+b)(x+y)-2xy}{2\sqrt{(x-a)(b-x)}\sqrt{(y-a)(b-y)}}dxdy,$$

with equality for $\phi(x) = c_1 + c_2x$.

On the Circle – in work

Theorem

If $V : S^1 \to \mathbb{R}$ *is a potential such that* $V(e^{ix}) - (\rho - 1/4)x^2$ *is convex on* $\mathbb{R}$*, then*

$$2\rho \iint \left| \frac{f(z) - f(w)}{z - w} \right|^2 \alpha(dz)\, \alpha(dw) \leq \int |f'|^2 d\mu_V$$

holds for any smooth $f : S^1 \to \mathbb{C}$*.*

Thank You!