Loop measures and loop soups

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Markov processes

$$\{X_t, t \geq 0\}$$

Markov process in S, with transition densities

$$p_t(x, y).$$

$$E^x(f(X_t)) = \int f(y)p_t(x, y)\, dm(y).$$

Potential densities

$$u(x,y)=\int_0^\infty p_t(x,y)\,dt.$$

$$u^\alpha(x,y)=\int_0^\infty e^{-\alpha t}p_t(x,y)\,dt.$$

Resolvent equation

$$u^{\alpha}(x,y)-u^{\beta}(x,y)$$
$$=(\beta-\alpha)\int u^{\alpha}(x,z)u^{\beta}(z,y)\,dm(z).$$

$$\frac{d}{d\alpha}u^{\alpha}(x,y)=-\int u^{\alpha}(x,z)u^{\alpha}(z,y)\,dm(z)$$

Basic martingale

$$M_s = p_{t-s}(X_s, x)$$

is a martingale, $0 \leq s \leq t$.

Basic martingale

$$M_s = p_{t-s}(X_s, x) = p_{t-s}(X_{s-r}, x) \circ \theta_r$$

hence

$$E^x\left(M_s \,|\, \mathcal{F}_r\right) = E^{X_r}\left(p_{t-s}(X_{s-r}, x)\right)$$

$$= \int p_{s-r}(X_r, z) p_{t-s}(z, x)\, dm(z) = M_r.$$

Bridge measure

Hence

$$Q_t^{x,x}\left(G\right) = P^x\left(G\, p_{t-s}(X_s, x)\right),$$

for all $G \in \mathcal{F}_s$ with $s < t$,

is well-defined.

Extends to $\mathcal{F}_t^-$.

Loop measure

$$\mu(F) = \int_0^\infty \frac{1}{t} \int Q_t^{x,x}\left(F \circ k_t\right)\, dm(x)\, dt$$

for all $F \in \mathcal{F}$, where k_t is the killing operator:

$$k_t\omega(s) = \omega(s), \text{ if } s < t$$

and

$$k_t\omega(s) = \Delta, \text{ if } s \geq t.$$

k_t example

$$\left(\int_0^\infty f(X_s)\,ds\right)\circ k_t = \int_0^t f(X_s)\,ds.$$

Loop measure moment formula

$$\mu\left(\prod_{j=1}^{k}\left(\int_0^\infty f_j(X_t)\,dt\right)\right)$$

$$= \sum_{\pi\in\mathcal{P}_k^{\odot}} \int u(y_1,y_2)\cdots u(y_{k-1},y_k)u(y_k,y_1)$$

$$\prod_{j=1}^{k} f_{\pi(j)}(y_j)\,dm(y_j)$$

Loop measure

$$\mu(F) = \int_0^\infty \frac{1}{t} \int Q_t^{x,x}\left(F \circ k_t\right) \, dm(x) \, dt$$

$$= \int_0^\infty \int_0^\infty e^{-\alpha t} \int Q_t^{x,x}\left(F \circ k_t\right) \, dm(x) \, dt \, d\alpha$$

Loop measure, 0

$$0 < t_1 < t_2 < \cdots < t_k$$

$$\prod_{j=1}^{k} f_j(X_{t_j}) \circ k_t = 1_{\{t > t_k\}} \prod_{j=1}^{k} f_j(X_{t_j})$$

Loop measure, 1

$$0 < t_1 < t_2 < \cdots < t_k < t$$

$$Q_t^{x,x}\left(\prod_{j=1}^k f_j(X_{t_j})\right)$$

$$= \int p_{t_1}(x,y_1)p_{t_2-t_1}(y_1,y_2)\cdots$$

$$\cdots p_{t_k-t_{k-1}}(y_{k-1},y_k)p_{t-t_k}(y_k,x)\prod_{j=1}^k f_j(y_j)\,dm(y_j)$$

Loop measure, 2

$$Q_t^{x,x}\left(\int_{\{0\le t_1\le\cdots\le t_k<\infty\}}\prod_{j=1}^k f_j(X_{t_j})\circ k_t\ \ dt_j\right)$$

$$=\int_{\{0\le t_1\le\cdots\le t_k<t\}}\int p_{t_1}(x,y_1)p_{t_2-t_1}(y_1,y_2)\cdots$$

$$\cdots p_{t_k-t_{k-1}}(y_{k-1},y_k)p_{t-t_k}(y_k,x)$$

$$\prod_{j=1}^k f_j(y_j)\,dm(y_j)\ \ dt_j$$

Loop measure, 2a

$$\int_0^\infty e^{-\alpha t} Q_t^{x,x} \left(\int_{\{R^k_\le\}} \prod_{j=1}^k f_j(X_{t_j}) \circ k_t \ \ dt_j \right) \ dt$$

$$= \int u^\alpha(x, y_1) u^\alpha(y_1, y_2) \cdots u^\alpha(y_{k-1}, y_k) u^\alpha(y_k, x)$$

$$\prod_{j=1}^k f_j(y_j) \, dm(y_j)$$

Recall

$$\int u^\alpha(y_k, x) u^\alpha(x, y_1)\, dm(x)$$
$$= -\frac{d}{d\alpha} u^\alpha(y_k, y_1).$$

Loop measure, 2b

$$\int \int_0^\infty e^{-\alpha t} Q_t^{x,x} \left(\int_{\{R_{\le}^k\}} \prod_{j=1}^k f_j(X_{t_j}) \circ k_t \right) \, dt \, dm(x)$$

$$= - \int u^\alpha(y_1, y_2) \cdots u^\alpha(y_{k-1}, y_k) \frac{d}{d\alpha} u^\alpha(y_k, y_1) \prod_{j=1}^k f_j(y_j) \, dm(y_j)$$

Loop measure, 2c

$$\int \int_0^\infty e^{-\alpha t} Q_t^{x,x} \left(\prod_{j=1}^k \int_0^\infty f_j(X_{t_j})\, dt_j \circ k_t \right) \, dt\, dm(x)$$

$$= - \int u^\alpha(y_1, y_2) \cdots u^\alpha(y_{k-1}, y_k) \frac{d}{d\alpha} u^\alpha(y_k, y_1)$$

$$\sum_{\pi \in \mathcal{P}_k} \prod_{j=1}^k f_{\pi(j)}(y_j)\, dm(y_j).$$

Loop measure, 2d

$$\int \int_0^\infty e^{-\alpha t} Q_t^{x,x} \left(\prod_{j=1}^{k} \int_0^\infty f_j(X_{t_j})\, dt_j \circ k_t \right) \, dt\, dm(x)$$

$$= -\frac{1}{k} \frac{d}{d\alpha} \int u^\alpha(y_1, y_2) \cdots u^\alpha(y_{k-1}, y_k) u^\alpha(y_k, y_1)$$

$$\sum_{\pi \in \mathcal{P}_k} \prod_{j=1}^{k} f_{\pi(j)}(y_j)\, dm(y_j).$$

Loop measure, 2e

$$\mu\left(\prod_{j=1}^{k}\int_0^\infty f_j(X_{t_j})\,dt_j\right)$$
$$=\frac{1}{k}\int u(y_1,y_2)\cdots u(y_{k-1},y_k)u(y_k,y_1)$$
$$\sum_{\pi\in\mathcal{P}_k}\prod_{j=1}^{k} f_{\pi(j)}(y_j)\,dm(y_j).$$

Loop measure moment formula

$$\mu\left(\prod_{j=1}^{k}\left(\int_0^\infty f_j(X_t)\,dt\right)\right)$$

$$= \sum_{\pi\in\mathcal{P}_k^{\odot}} \int u(y_1,y_2)\cdots u(y_{k-1},y_k)u(y_k,y_1)$$

$$\prod_{j=1}^{k} f_{\pi(j)}(y_j)\,dm(y_j)$$

Conditions

$$\sup_x \int \left(u^2(x,y) + u^2(y,x)\right) |f(y)|\, dm(y) < \infty$$

$$\sup_x \int u(x,y)|f(y)|\, dm(y) < \infty$$

$$\int |f(y)|\, dm(y) < \infty.$$

Special case

Assume that $u(x,y)$ is continuous.

$$L_t^x = \lim_{\epsilon \to 0} \int_0^t f_{\epsilon,x}(X_r)\, dr.$$

Local time at x.

$$L_\infty^x = \lim_{\epsilon \to 0} \int_0^\infty f_{\epsilon,x}(X_r)\, dr.$$

Local time moment formula

$$\mu\left(\prod_{j=1}^{k} L_\infty^{x_j}\right)$$
$$= \sum_{\pi\in\mathcal{P}_k^{\odot}} u(x_{\pi(1)}, x_{\pi(2)})u(x_{\pi(2)}, x_{\pi(3)})\cdots$$
$$\cdots u(x_{\pi(k-1)}, x_{\pi(k)})u(x_{\pi(k)}, x_{\pi(1)}).$$

one cycle

Local time moment formula, 2

Or:

$$\mu\left(L^x_\infty \prod_{j=1}^k L^{x_j}_\infty\right)$$
$$= \sum_{\pi\in\mathcal{P}_k} u(x, x_{\pi(1)})u(x_{\pi(1)}, x_{\pi(2)}) \cdots$$
$$\cdots u(x_{\pi(k-1)}, x_{\pi(k)})u(x_{\pi(k)}, x).$$

X symmetric: $p_t(x,y)$ is positive definite

$$\sum_{i,j=1}^{n} a_i a_j\, p_t(x_i, x_j)$$

$$= \sum_{i,j=1}^{n} a_i a_j \int p_{t/2}(x_i, z)\, p_{t/2}(z, x_j)\, dm(z)$$

$$= \int |\sum_{i=1}^{n} a_i\, p_{t/2}(x_i, z)|^2\, dm(z) \geq 0.$$

X symmetric: potential $u(x,y)$ is positive definite

Hence

$$u(x,y) =: \int_0^\infty p_t(x,y)\,dt$$

is symmetric and positive definite.

We say that the Gaussian process

$$G := \{G_x, x \in S\}$$

with

$$E\left(G_x G_y\right) = u(x,y) =: \int_0^\infty p_t(x,y)\,dt$$

is associated with X.

Continuity, symmetric MP

Local time for X:

$$L_t^x = \lim_{\epsilon \to 0} \int_0^t f_{\epsilon,x}(X_r)\, dr.$$

Theorem

$$\{L_t^x, (x,t) \in S \times R_+^1\}$$

is continuous if and only if the associated Gaussian process is continuous.

$$\{G_x, x \in S\}$$

is continuous a.s. if and only if there exists a probability measure μ on S

$$\lim_{\delta \to 0} \sup_{s \in S} \int_0^\delta \left(\log \frac{1}{\mu(B_d(s,u))} \right)^{1/2} du = 0.$$

$$d(x,y) = \left(E\left(\{G_x - G_y\}^2 \right) \right)^{1/2}.$$

Dynkin Isomorphism Theorem

$$E_G Q^{x,x}\left(F\left(L_\infty^{x_i}+\frac{1}{2}G_{x_i}^2\right)\right)$$
$$=E_G\left(G_x^2\,F\left(\frac{1}{2}G_{x_i}^2\right)\right).$$

$$Q^{z,x}(F)=\int_0^\infty Q_t^{z,x}\left(F\circ k_t\right)\,dt.$$

$$Q^{x,x}(1)=u(x,x)<\infty.$$

$Q^{z,x}$ moment formula

$$Q^{z,x}\left(\prod_{j=1}^{k}\left(\int_0^\infty f_j(X_t)\,dt\right)\right)$$

$$=\sum_{\pi\in\mathcal{P}_k}\int u(z,y_1)u(y_1,y_2)\cdots$$

$$\cdots u(y_{k-1},y_k)u(y_k,x)\prod_{j=1}^{k} f_{\pi(j)}(y_j)\,dm(y_j).$$

$Q^{z,x}$ moment formula, 2

$$Q^{z,x}\left(\prod_{j=1}^{k} L_{\infty}^{x_j}\right)$$
$$= \sum_{\pi\in\mathcal{P}_k} u(x, x_{\pi(1)})u(x_{\pi(1)}, x_{\pi(2)})\cdots$$
$$\cdots u(x_{\pi(k-1)}, x_{\pi(k)})u(x_{\pi(k)}, x).$$

$Q^{z,x}$ moment formula, 3

$$\mu\left(L^x_\infty \prod_{j=1}^k L^{x_j}_\infty\right)$$

$$= \sum_{\pi\in\mathcal{P}_k} u(x, x_{\pi(1)})u(x_{\pi(1)}, x_{\pi(2)})\cdots$$

$$\cdots u(x_{\pi(k-1)}, x_{\pi(k)})u(x_{\pi(k)}, x)$$

$$= Q^{x,x}\left(\prod_{j=1}^k L^{x_j}_\infty\right).$$

h-transform

$$Q^{z,x}(F1_{t<\zeta}) = E^z(F\,u(X_t, x)), \quad F \in \mathcal{F}_t$$

Loop soup

Let $\mathcal{L}_\alpha$ be a Poisson point process on Ω_Δ with intensity measure $\alpha\mu$. Thus if

$$N(A) := \#\{\mathcal{L}_\alpha \cap A\}, \quad A \subseteq \Omega_\Delta.$$

Then

$$P\left(N(A) = k\right) = \frac{(\alpha\mu(A))^k}{k!} e^{-\alpha\mu(A)}.$$

The master formula for Poisson processes

$$E\left(e^{\sum_{\omega\in\mathcal{L}_\alpha} h(\omega)}\right)$$
$$=\exp\left(\alpha\left(\int_{\Omega_\Delta}\left(e^{h(\omega)}-1\right)\,d\mu(\omega)\right)\right)$$

The Palm formula

$$E_{\mathcal{L}_\alpha}\left(\left(\sum_{\omega\in\mathcal{L}_\alpha} h(\omega)\right) G(\mathcal{L}_\alpha)\right)$$
$$= \alpha \int E_{\mathcal{L}_\alpha}\left(G(\omega' \cup \mathcal{L}_\alpha)\right) h(\omega')\, d\mu(\omega').$$

To the Isomorphism Theorem

$$\widehat{L}^x = \sum_{\omega \in \mathcal{L}_\alpha} L^x_\infty(\omega).$$

Take $h(\omega) = L^x_\infty(\omega)$, $G(\mathcal{L}) = F\left(\widehat{L}^{x_j}\right)$, and note

$$\widehat{L}^{x_j}(\omega' \cup \mathcal{L}_\alpha) = \sum_{\omega \in \omega' \cup \mathcal{L}_\alpha} L^{x_j}_\infty(\omega)$$
$$= \widehat{L}^{x_j}(\mathcal{L}_\alpha) + L^{x_j}_\infty(\omega'),$$

To the Isomorphism Theorem, 2

Thus

$$G(\omega' \cup \mathcal{L}) = F\left(\widehat{L}^{x_j}(\mathcal{L}_\alpha) + L_\infty^{x_j}(\omega')\right).$$

and then by Palm

$$E_{\mathcal{L}_\alpha}\left(\widehat{L}^x F\left(\widehat{L}^{x_j}\right)\right)$$
$$= \alpha E_{\mathcal{L}_\alpha}\int\left(L_\infty^x(\omega') F\left(\widehat{L}^{x_j} + L_\infty^{x_j}(\omega')\right) d\mu(\omega')\right)$$

Local time moment formula, 2a

$$\mu\left(L_\infty^x \prod_{j=1}^k L_\infty^{x_j}\right)$$
$$= \sum_{\pi\in\mathcal{P}_k} u(x, x_{\pi(1)})u(x_{\pi(1)}, x_{\pi(2)})\cdots$$
$$\cdots u(x_{\pi(k-1)}, x_{\pi(k)})u(x_{\pi(k)}, x)$$
$$= Q^{x,x}\left(\prod_{j=1}^k L_\infty^{x_j}\right)$$

To the Isomorphism Theorem, 3

$$E_{\mathcal{L}_\alpha}\left(\widehat{L}^x F\left(\widehat{L}^{x_j}\right)\right)$$

$$= \alpha E_{\mathcal{L}_\alpha} \int \left(L^x_\infty(\omega') F\left(\widehat{L}^{x_j} + L^{x_j}_\infty(\omega')\right) \, d\mu(\omega')\right)$$

$$= \alpha E_{\mathcal{L}_\alpha} Q^{x,x}\left(F\left(\widehat{L}^{x_j} + L^{x_j}_\infty\right)\right)$$

The Isomorphism Theorem?

$$E_{\mathcal{L}_\alpha}\left(\widehat{L}^x\, F\left(\widehat{L}^{x_j}\right)\right) = \alpha E_{\mathcal{L}_\alpha} Q^{x,x}\left(F\left(\widehat{L}^{x_j} + L_\infty^{x_j}\right)\right)$$

Recall

$$E_G\left(G_x^2\, F\left(\frac{1}{2}G_{x_j}^2\right)\right) = E_G Q^{x,x}\left(F\left(\frac{1}{2}G_{x_j}^2 + L_\infty^{x_j}\right)\right).$$

What is $\widehat{L}^x$?

Loop soups and permanental processes, A

$$\widehat{L}^x = \sum_{\omega \in \mathcal{L}_\alpha} L^x_\infty(\omega).$$

$$E_{\mathcal{L}_\alpha}\left(e^{\sum_{j=1}^n z_j \widehat{L}^{x_j}}\right) = \exp\left(\alpha\left(\int_{\Omega_\Delta}\left(e^{\sum_{j=1}^n z_j L^{x_j}_\infty(\omega)} - 1\right) d\mu(\omega)\right)\right)$$

Loop soups and permanental processes, B

$$E_{\mathcal{L}_\alpha}\left(e^{\sum_{j=1}^n z_j \widehat{L}^{x_j}}\right) = \exp\left(\alpha\left(\int_{\Omega_\Delta}\left(e^{\sum_{j=1}^n z_j L_\infty^{x_j}(\omega)} - 1\right)\, d\mu(\omega)\right)\right)$$

$$E_{\mathcal{L}_\alpha}\left(\widehat{L}^{x_1} e^{\sum_{j=2}^n z_j \widehat{L}^{x_j}}\right) = \alpha\left(\int_{\Omega_\Delta} L_\infty^{x_1} e^{\sum_{j=2}^n z_j L_\infty^{x_j}(\omega)}\, d\mu(\omega)\right)$$

$$\exp\left(\alpha\left(\int_{\Omega_\Delta}\left(e^{\sum_{j=2}^n z_j L_\infty^{x_j}(\omega)} - 1\right)\, d\mu(\omega)\right)\right).$$

Loop soups and permanental processes, C

$$\widehat{L}^x = \sum_{\omega \in \mathcal{L}_\alpha} L^x_\infty(\omega).$$

$$E_{\mathcal{L}_\alpha}\left(\prod_{j=1}^{n} \widehat{L}^{x_j}\right) = \sum_{\ell=1}^{n} \sum_{\cup_{i=1}^{\ell} B_i = [1,n]} \alpha^\ell \prod_{i=1}^{\ell} \mu\left(\prod_{j \in B_i} L^{x_j}_\infty\right).$$

Loop soups and permanental processes, 2

If $B_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,|B_i|}\}$,

$$E_{\mathcal{L}_\alpha}\left(\prod_{j=1}^{n} \widehat{L}^{x_j}\right) = \sum_{\ell=1}^{n} \sum_{\cup_{i=1}^{\ell} B_i = [1,n]}$$

$$\alpha^\ell \prod_{i=1}^{\ell} \sum_{\pi \in \mathcal{P}^{\odot}_{|B_j|}} u(x_{i,\pi_1}, x_{i,\pi_2}) \cdots u(x_{i,\pi_{|B_i|}}, x_{i,\pi_1}).$$

many cycles

Loop soups and permanental processes, 3

Thus

$$E_{\mathcal{L}_\alpha}\left(\prod_{j=1}^{n}\widehat{L}^{x_j}\right)$$
$$=\sum_{\pi}\alpha^{c(\pi)}\,u(x_1,x_{\pi_1})u(x_2,x_{\pi_2})\cdots u(x_n,x_{\pi_n}).$$

Towards the Gaussian moment formula

$$E\left(e^{-\sum_{j=1}^{m} z_j G_{x_j}}\right) = e^{E\left(\sum_{j=1}^{m} z_j G_{x_j}\right)^2/2}$$

$$= e^{\sum_{j,k=1}^{m} z_j z_k u(x_j,x_k)/2}.$$

Towards the Gaussian moment formula

$$E\left(e^{-\sum_{j=1}^m z_j G_{x_j}}\right) = e^{E\left(\sum_{j=1}^m z_j G_{x_j}\right)^2/2}$$
$$= e^{\sum_{j,k=1}^m z_j z_k u(x_j,x_k)/2}.$$

$$E\left(G_{x_1} e^{-\sum_{j=2}^m z_j G_{x_j}}\right) = e^{\sum_{j,k=2}^m z_j z_k u(x_j,x_k)/2}$$
$$\times(-1)\sum_{k=2}^m z_k u(x_1,x_k)$$

Gaussian moment formula, 1

$$E\left(\prod_{j=1}^{2k} G_{x_j}\right)$$

$$= \sum_{\text{pairings}} \prod_{j=1}^{k} u(x_{p_j(1)}, x_{p_j(2)}).$$

$$\{1,\ldots,2k\} = \cup_{j=1}^{k}\{p_j(1), p_j(2)\}$$

Gaussian moment formula, 2

$$E\left(\prod_{j=1}^{k} G_{x_j}^2\right) = \sum_{B_1,\ldots,B_l} 2^{k-l} U(B_1)\cdots U(B_l)$$

$$U(B_j) = \sum_{\pi\in\mathcal{P}^{\odot}_{|B_j|}} \prod_{i\in B_j} u(x_{\pi(i)}, x_{\pi(i^+)}).$$

many cycles

Key fact even without symmetry

For any pair x, y,

$$\{\widehat{L}^{x}_{1/2}, \widehat{L}^{y}_{1/2}\} \stackrel{law}{=} \{G^2(x)/2, G^2(y)/2\}$$

where $\{G(x), G(y)\}$ is a mean zero Gaussian random vector with covariance matrix given by

$$E\left(G(x)G(y)\right) = \left(u(x,y)u(y,x)\right)^{1/2}.$$

Orlicz norms

Let

$$\psi_2(x) = \exp(x^2) - 1$$

and $L^{\psi_2}(\Omega, \mathcal{F}, P)$ denote the set of random variables $\xi : \Omega \to R^1$ such that $E\psi_2\left(|\xi|/c\right) < \infty$ for some $c > 0$. $L^{\psi_2}(\Omega, \mathcal{F}, P)$ is a Banach space with norm given by

$$\|\xi\|_{\psi_2} = \inf\left\{c > 0 : E\psi_2\left(|\xi|/c\right) \leq 1\right\}.$$

Orlicz norm of a normal random variable

If G is normal

$$\|G\|_{\psi_2} = \frac{2}{\sqrt{3/2}} \|G\|_2.$$

Proof of Continuity

$$\{\left(\widehat{L}^x\right)^{1/2}, \left(\widehat{L}^y\right)^{1/2}\} \overset{law}{=} \{|G_x|, |G_y|\}$$

so that

$$\| \left(\widehat{L}^x\right)^{1/2} - \left(\widehat{L}^y\right)^{1/2} \|_{\psi_2} = \| \, |G_x| - |G_y| \, \|_{\psi_2}$$

$$\leq \| \, G_x - G_y \, \|_{\psi_2} = d(x, y).$$

Proof of Continuity

Chaining argument requires a metric. Use

$$\widetilde{d}(x,y) = \| \left(\widehat{L}^x\right)^{1/2} - \left(\widehat{L}^y\right)^{1/2} \|_{\psi_2}$$

$$\leq d(x,y).$$

For a general, not necessarily symmetric Markov process, the local time

$$\{L_t^x, (x,t) \in S \times R_+^1\}$$

is continuous if the associated permanental process is continuous.

(Open problem: and only if?)

inverse local time, $u < \infty$

$$\tau(t) = \inf\{s \,|\, L_s^0 > t\}$$

When X is recurrent, $\tau(t) \uparrow \infty$ as $t \uparrow \infty$.

Dynkin IT vs. RK2

$$E_G Q^{x,y}\left(F\left(L^{x_i}_\infty + \frac{1}{2}G^2_{x_i}\right)\right)$$
$$= E_G\left(F\left(\frac{1}{2}G^2_{x_i}\right)G_x G_y\right).$$

$$E_G P^0\left(F\left(L^{x_i}_{\tau(t)} + \frac{1}{2}\eta^2_{x_i}\right)\right)$$
$$= E_G\left(F\left(\frac{1}{2}(\eta_{x_i} + \sqrt{2t})^2\right)\right).$$

Open Problem

$$E_G P^0 \left(F \left(L^{x_i}_{\tau(t)} + \frac{1}{2}\eta^2_{x_i} \right) \right)$$
$$= E_G \left(F \left(\frac{1}{2}(\eta_{x_i} + \sqrt{2t})^2 \right) \right).$$

What is the analogue for non-symmetric Markov process?

Generalizations: no transition densities

$$\int F\,d\mu = \int_S P^x \left(\int_0^\infty \frac{1}{t} F \circ k_t \; d_t L_t^x \right) \, dm(x).$$

Compare

$$Q^{x,x}(F) = P^x \left(\int_0^\infty F \circ k_t \; d_t L_t^x \right).$$

Recall general moment formula

$$\mu\left(\prod_{j=1}^{k}\left(\int_0^\infty f_j(X_t)\,dt\right)\right)$$
$$=\sum_{\pi\in\mathcal{P}_k^{\odot}}\int u(y_1,y_2)\cdots u(y_{k-1},y_k)u(y_k,y_1)$$
$$\prod_{j=1}^{k} f_{\pi(j)}(y_j)\,dm(y_j)$$

Caution!

$$\mu\left(\int_0^\infty f(X_t)\,dt\right) = \int u(y_1, y_1) f(y_1)\,dm(y_1) = \infty$$

if

$$u(y, y) = \infty, \quad \forall y.$$

But

$$\text{But} \quad Q_t^{x,x}\left(\int_0^t f(X_s)\,ds\right)$$

$$= \int \int_0^t p_s(x,y)p_{t-s}(y,x)\,ds\, f(y)\,dm(y)$$

so that

$$\int Q_t^{x,x}\left(\int_0^t f(X_s)\,ds\right)\,dm(x)$$

$$= t\int p_t(y_1,y_1)f(y_1)\,dm(y_1).$$

But 3

Thus if we set

$$\phi(f) = \int_0^\infty f(X_t)\, dt,$$

then

$$\mu\left(1_{\{\zeta>\delta\}}\phi(f)\right)$$
$$= \int \left(\int_\delta^\infty p_t(y_1, y_1)\, dt\right) f(y_1)\, dm(y_1)$$

Loop soup field

$$\psi(f) = \lim_{\delta \to 0} \widehat{\phi}_\delta(f),$$

where

$$\widehat{\phi}_\delta(f) = \left(\sum_{\omega \in \mathcal{L}_\alpha} 1_{\{\zeta(\omega) > \delta\}} \phi(f)(\omega) \right)$$

$$-\alpha\mu(1_{\{\zeta > \delta\}} \phi(f)).$$

Permanental fields, 1

By the master formula

$$E_{\mathcal{L}_\alpha}\left(e^{\sum_{j=1}^n z_j\widehat{\phi}_\delta(f_j)}\right) =$$

$$= \exp\left(\alpha\left(\int_{\Omega_\Delta}\left(e^{\sum_{j=1}^n z_j 1_{\{\zeta>\delta\}}\phi(f_j)} - \sum_{j=1}^n z_j 1_{\{\zeta>\delta\}}\phi(f_j) - 1\right)\, d\mu(\omega)\right)\right)$$

Permanental fields, 2

$$E_{\mathcal{L}_\alpha}\left(\prod_{j=1}^{n}\widehat{\phi}_\delta(f_j)\right)$$

$$=\sum_{\cup_i B_i=[1,n],\,|B_i|\geq 2}\ \prod_i \alpha\ \mu\left(\prod_{j\in B_i} 1_{\{\zeta>\delta\}}\phi(f_j)\right)$$

Permanental fields, 3

$$E_{\mathcal{L}_\alpha}\left(\prod_{j=1}^{n}\psi(f_j)\right)$$

$$=\sum_{\cup_i B_i=[1,n],\,|B_i|\geq 2}\prod_i \alpha\;\mu\left(\prod_{j\in B_i}\phi(f_j)\right)$$

Permanental fields, 4

$$E_{\mathcal{L}_\alpha}\left(\prod_{j=1}^{n}\psi(f_j)\right)$$

$$=\sum_{\pi\in\mathcal{P}'}\alpha^{c(\pi)}\int\prod_{j=1}^{n}u(x_j,x_{\pi(j)})$$

$$\prod_{j=1}^{n}f_j(x_j)\,dm(x_j).$$

Symmetric case

Gausian field $G(f)$ with

$$E\left(G(f)G(f')\right)$$

$$= \int u(x,y)f(x)f'(y)\,dm(x)\,dm(y).$$

Symmetric case, 2

Wick square $:G^2:(f)$ with

$$E\left(:G^2:(f)\quad :G^2:(f')\right)$$

$$=\int u^2(x,y)f(x)f'(y)\,dm(x)\,dm(y).$$

$$:G^2:(f)$$

$$=\lim_{\delta\to 0}\int \left(G^2_{x,\delta}-E(G^2_{x,\delta})\right)\; f(x)\,dm(x).$$

Symmetric case, 2

$$\psi(f) =: G^2 : (f)/2.$$

To the Isomorphism Theorem

$$Q_g^h(F) = \int Q^{x,x}\left(F \int_0^\infty g(X_t)\,dt\right) h(x)\,dm(x)$$

$$Q_g^h(1) = \int Q^{x,x}\left(\int_0^\infty g(X_t)\,dt\right) h(x)\,dm(x)$$

$$= \int u(x,y)u(y,x)g(y)h(x)\,dm(y)\,dm(x)$$

Finite!

The Isomorphism Theorem

$$E_{\mathcal{L}_\alpha} Q_g^h \left(F\left(\psi(f_j) + \phi(f_j)\right)\right) = \frac{1}{\alpha} E_{\mathcal{L}_\alpha}\left(\theta^{g,h} F\left(\psi(f_j)\right)\right)$$

where

$$\theta^{g,h} = \sum_{\omega \in \mathcal{L}_\alpha} \phi(g)(\omega)\ \phi(h)(\omega)$$

has all moments finite.

To the Isomorphism Theorem

Take $h(\omega) = \phi(f)(\omega)$, $G(\mathcal{L}) = F\left(\psi(f_j)\right)$,
and note

$$\psi(f_j)(\omega' \cup \mathcal{L}_\alpha) = \lim_{\delta \to 0} \left(\sum_{\omega \in \omega' \cup \mathcal{L}_\alpha} 1_{\{\zeta(\omega) > \delta\}} \phi(f)(\omega) \right.$$

$$\left. - \alpha\mu(1_{\{\zeta > \delta\}} \phi(f)\right).$$

$$= \psi(f_j)(\mathcal{L}_\alpha) + \phi(f_j)(\omega')$$

To the Isomorphism Theorem, 2

Thus

$$G(\omega' \cup \mathcal{L}) = F\left(\psi(f_j)(\mathcal{L}_\alpha) + \phi(f_j)(\omega')\right),$$

and then by Palm

$$E_{\mathcal{L}_\alpha}\left(\widehat{\phi}(f)\, F\left(\psi(f_j)\right)\right)$$

$$= \alpha E_{\mathcal{L}_\alpha} \int \left(\phi(f)(\omega')\, F\left(\psi(f_j) + \phi(f_j)(\omega')\right)\, d\mu(\omega')\right)$$

Generalizations: CAF's

$$L_t^{\nu} = \lim_{\epsilon \to 0} \int_S \int_0^t f_{\epsilon,x}(X_r)\, dr\, d\nu(x).$$

Loop measure moment formula for CAF's

$$\mu\left(\prod_{j=1}^{k} L_\infty^{\nu_j}\right)$$

$$= \sum_{\pi \in \mathcal{P}_k^{\odot}} \int u(y_1, y_2) \cdots u(y_{k-1}, y_k) u(y_k, y_1)$$

$$\prod_{j=1}^{k} d\nu_{\pi(j)}(x_j)$$

Intersection local times

Let

$$L(x,\epsilon) := \int_0^\infty f_{x,\epsilon}(X_t)\, dt.$$

$L(x,\epsilon)$ can be thought of as the approximate total local time of X at the point $x \in R^d$.

Intersection local times, 2

$$L_2(x,\epsilon) = L^2(x,\epsilon) - 2\,\mathrm{ch}(\epsilon) L(x,\epsilon)$$

where

$$\mathrm{ch}(\epsilon) = \int u(\epsilon y_1, \epsilon y_2) \prod_{j=1}^{2} f(y_j)\, dy_j$$

Intersection local times, 3

$$L_{2,\epsilon}(g) = \int L_2(x,\epsilon) g(x)\, dm(x).$$

Then

$$L_2(g) = \lim_{\epsilon \to 0} L_{2,\epsilon}(g)$$

exists in $L^p(\mu)$, for all $p \geq 2$.

loop soup self-intersection local time

$$\psi_2(g) = \lim_{m\to\infty} \psi_{2,m}(g),$$

where

$$\psi_{2,m}(g) = \left(\sum_{\omega\in\mathcal{L}_\alpha} 1_{D_m} L_2(g)(\omega) \right) - \alpha\mu(1_{D_m} L_2(g))$$

Permanental Wick power

$$L_1(x,r)=\int_0^\infty f_{x,r}(X_t)\,dt.$$

Set

$$\psi(x,r)$$

$$=\lim_{m\to\infty}\left(\sum_{\omega\in\mathcal{L}_\alpha}1_{D_m}L_1(x,r)(\omega)\right)-\alpha\mu(1_{D_m}L_1(x,r)).$$

Permanental Wick power, 2

$$\widetilde{\psi}_2(x,\epsilon) = \psi^2(x,\epsilon) - 2\mathrm{ch}_1(\epsilon)\psi(x,\epsilon) - \alpha\mathrm{ci}_2(\epsilon)$$

where

$$\mathrm{ci}_2(\epsilon)$$

$$= \int u(\epsilon y_1, \epsilon y_2) u(\epsilon y_k, \epsilon y_1) \prod_{j=1}^{2} f(y_j)\, dy_j$$

Comparison

$$\widetilde{\psi}_2(g) = \psi_2(g) + \mathcal{I}_{1,1}(g)$$

and formally

$$\mathcal{I}_{1,1}(g) = \sum_{\omega \neq \omega'} \int L_1(x,r)(\omega) L_1(x,r)(\omega') g(x)\, dm(x)$$

$$= \int \int_0^\infty \int_0^\infty f_r(\omega_s - x)\, f_r(\omega'_t - x)\, ds\, dt\, g(x)\, dm(x).$$