In Memory of Wenbo V Li's Contributions

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Outline

- Lower tail probabilities
- Capture time of Brownian pursuits
- Random polynomials
- Normal comparison inequalities
- Wenbo's weak correlation inequality

1. Lower tail probabilities

Let $\{X_t,\ t \in T\}$ be a real valued Gaussian process indexed by T with $\mathbb{E}\, X_t = 0$. Lower tail probability refers to

$$\mathbb{P}\left(\sup_{t \in T} (X_t - X_{t_0}) \le x \right) \text{ as } x \to 0,\ t_0 \in T$$

or

$$\mathbb{P}\left(\sup_{t \in T} X_t \le x \right) \text{ as } |T| \to \infty$$

- Examples:
  - (a) Capture time of Brownian pursuits
  - (b) Random polynomials

- Li and Shao (2004): Established a general result for the lower tail probability of non-stationary Gaussian process

$$\mathbb{P}\left(\sup_{t \in T} (X_t - X_{t_0}) \leq x \right) \text{ as } x \to 0,$$

Special cases:

- *Let* $\{X(t), t \in [0,1]^d\}$ *be a centered Gaussian process with* $X(0) = 0$ *and stationary increments, that is*

$$\forall\, t, s \in [0,1]^d, \;\; \mathbb{E}\,(X_t - X_s)^2 = \sigma^2(||t - s||).$$

If there are $0 < \alpha \le \beta < 1$ *such that*

$$\sigma(h)/h^{\alpha} \uparrow, \;\; \sigma(h)/h^{\beta} \downarrow$$

Then there exist $0 < c_1 \le c_2 < \infty$ *such that for* $0 < x < 1/2$

$$-c_2 \ln \frac{1}{x} \le \ln \mathbb{P}\Big(\sup_{t \in [0,1]^d} X(t) \le \sigma(x) \Big) \le -c_1 \ln \frac{1}{x}.$$

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In particular, for the fractional Levy's Brownian motion $L_\alpha(t)$ of order α, i.e. $L_\alpha(0) = 0$ and

$$\mathbb{E}\,(L_\alpha(t) - L_\alpha(s))^2 = ||t-s||^{2\alpha},$$

$$\ln \mathbb{P}\Big(\sup_{t \in [0,1]^d} L_\alpha(t) \le x \Big) \approx -\ln \frac{1}{x}.$$

- Let $\{X(t), t \in [0,1]^d\}$ be a centered Gaussian process with $X(0) = 0$ and

$$\mathbb{E}\left(X_t X_s\right) = \prod_{i=1}^{d} \frac{1}{2}(\sigma^2(t_i) + \sigma^2(s_i) - \sigma^2(|t_i - s_i|)).$$

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In particular, for d-dimensional fractional Brownian sheet $B_\alpha(t)$ (i.e., $\sigma(h) = h^\alpha$)

$$\ln \mathbb{P}\left(\sup_{t \in [0,1]^d} B_\alpha(t) \le x \right) \approx -\ln^d \frac{1}{x}$$

- Li and Shao (2004):

  Obtained a connection for the lower tail probabilities between a non-stationary Gaussian process and a dual-stationarized Gaussian processes.

▶ Explicit bounds of lower tail probabilities (Li and Xiao (2013))

Let $B(t)$ be the Brownian motion.

- For $0 < \theta < 1$

$$P(\sup_{0\le t\le 1}(B(t)-\theta B(1)) \le x) \sim \frac{1}{3\theta^2(1-\theta)^2\sqrt{2\pi}}x^3,$$

$$P(\sup_{0\le t\le 1}(B(t)-\theta t B(1)) \le x) \sim (1-\theta)\sqrt{2/\pi}\, x$$

and

$$\ln P(\sup_{0\le t\le 1} B(t) - \int_0^1 B(s)ds \le x) \sim -x^2\, c$$

where c is a specified constant.

2. Capture time of Brownian pursuits

Let $B_0, B_1, \cdots, B_n$ be independent standard Brownian motions. Define

$$\tau_n = \inf\left\{t > 0 : \max_{1\le k\le n} B_k(t) = B_0(t) + 1\right\}.$$

When is $\mathbb{E}(\tau_n)$ finite?

2. Capture time of Brownian pursuits

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When is $\mathbb{E}(\tau_n)$ finite?

Note that for any $a > 0$, by Brownian scaling,

$$\begin{aligned} &\mathbb{P}(\tau_n > t) \\ &= \mathbb{P}\Big(\max_{1 \le k \le n} \sup_{0 \le s \le t} (B_k(s) - B_0(s)) < 1 \Big) \\ &= \mathbb{P}\Big(\max_{1 \le k \le n} \sup_{0 \le s \le 1} (B_k(s) - B_0(s)) < t^{-1/2} \Big). \end{aligned}$$

Thus, the estimate is reduced to a lower tail probability problem.

- DeBlassie (1987):

$$\mathbb{P}\{\tau_n > t\} \sim ct^{-\gamma_n} \quad \text{as} \quad t \to \infty.$$

- Bramson and Griffeath (1991): $\mathbb{E}\,\tau_3 = \infty$
- Li and Shao (2001): $\mathbb{E}\,\tau_5 < \infty$.
- Ratzkin and Treibergs (2009): $\mathbb{E}\,\tau_4 < \infty$.

What is the asymptotic behavior of γ_n?

- Kesten (1992):

$$0 < \liminf_{n\to\infty} \gamma_n / \ln n \le \limsup_{n\to\infty} \gamma_n / \ln n \le 1/4$$

Conjecture: $\lim_{n\to\infty} \gamma_n / \ln n$ exists.

- Li and Shao (2002):

$$\lim_{n\to\infty} \gamma_n / \ln n = 1/4$$

Li and Shao (2002) also consider the capture time of the fractional Brownian motion pursuit.

Let $\{B_{k,\alpha}(t);\ t \geq 0\}(k = 0, 1, 2, \ldots, n)$ be independent fractional Brownian motions of order $\alpha \in (0, 1)$. Put

$$\tau_n := \tau_{n,\alpha} = \inf \left\{ t > 0 : \max_{1 \leq k \leq n} B_{k,\alpha}(t) = B_{0,\alpha}(t) + 1 \right\} .$$

Let

$$X_{k,\alpha}(t) = e^{-t\alpha} B_{k,\alpha}(e^t),\ k = 0, 1, \cdots, n$$

and

$$\gamma_{n,\alpha} := -\lim_{T \to \infty} \frac{1}{T} \ln \mathbb{P}\Big(\sup_{0 \leq t \leq T} \max_{1 \leq k \leq n} X_{k,\alpha}(t) \leq 0 \Big)$$

Li and Shao (2002):

$$\frac{1}{d_\alpha} \le \liminf_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} \le \limsup_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} < \infty,$$

where $d_\alpha = 2\int_0^\infty (e^{2\alpha x} + e^{-2\alpha x} - (e^x - e^{-x})^{2\alpha})dx$.

Li and Shao (2002):

$$\frac{1}{d_\alpha} \le \liminf_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} \le \limsup_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} < \infty,$$

where $d_\alpha = 2\int_0^\infty (e^{2\alpha x} + e^{-2\alpha x} - (e^x - e^{-x})^{2\alpha})dx.$

Conjecture:

$$\lim_{n\to\infty} \frac{\gamma_{n,\alpha}}{\ln n} = \frac{1}{d_\alpha}.$$

3. The probability that a random polynomial has no real root

Dembo, Poonen, Shao and Zeitouni (2002)

$$\mathbb{P}\Big(\sum_{i=0}^{n} Z_i x^i < 0 \ \forall\, x \in \mathbb{R}^1\Big) = n^{-b+o(1)}$$

where n is even, Z_i are i.i.d. $N(0,1)$, and

$$b = -4 \lim_{T\to\infty} \frac{1}{T} \ln \mathbb{P}\Big(\sup_{0\le t\le T} X(t) \le 0\Big)$$

where $X(t)$ is a centered stationary Gaussian process with

$$\mathbb{E}\, X(s)X(t) = \frac{2e^{-|t-s|/2}}{1+e^{-|t-s|}}$$

- Dembo, Poonen, Shao and Zeitouni (2002): $0.4 < b < 1.29$.
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- Li and Shao (2002): $0.5 < b < 1$

Let $\{B(t), t \geq 0\}$ be the Brownian motion and put $B_0(t) = B(t)$,

$$B_m(t) = \int_0^t B_{m-1}(s)ds$$

Li and Shao (2003+)

$$P(\sup_{0 \leq t \leq 1} B_m(t) \leq x) = x^{r_m + o(1)}$$

and

$$b \leq 2r_m(2m+1), \;\; 2r_m(2m+1) \to b$$

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Open questions:

- What is the value of b?
- If $\{X(t), t \geq 0\}$ is a differentiable stationary Gaussian process with positive correlation, what is the limit

$$\lim_{T \to \infty} \frac{1}{T} \ln P\Big(\sup_{0 \leq t \leq T} X(t) \leq 0\Big) \; ?$$

Let $n \geq 2$, and let $(\xi_j, 1 \leq j \leq n)$ be standard normal random variables with correlation matrix $R^1 = (r^1_{ij})$, let $(\eta_j, 1 \leq j \leq n)$ be standard normal random variables with correlation matrix $R^0 = (r^0_{ij})$. Set $\rho_{ij} = \max(|r^1_{ij}|, |r^0_{ij}|)$.

Slepian's Lemma: If $r^1_{ij} \geq r^0_{ij}$, then

$$\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \leq u_j\}\Big) \geq \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \leq u_j\}\Big)$$

Normal comparison inequality:

- Berman (1964,1971), Cramér and Leadbetter (1967)):

$$\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \le u_j\}\Big) - \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \le u_j\}\Big)$$
$$\le \frac{1}{2\pi}\sum_{1\le i<j\le n}(r_{ij}^1 - r_{ij}^0)^+ (1-\rho_{ij}^2)^{-1/2}\exp\Big(-\frac{u_i^2+u_j^2}{2(1+\rho_{ij})}\Big)$$

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- Li and Shao (2002):

$$\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \le u_j\}\Big) - \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \le u_j\}\Big)$$

$$\le \quad \frac{1}{4}\sum_{1\le i<j\le n}(r_{ij}^1 - r_{ij}^0)^+ \exp\Big(-\frac{u_i^2+u_j^2}{2(1+\rho_{ij})}\Big)$$

- Li and Shao (2002): If

$$r_{ij}^1 \geq r_{ij}^0 \geq 0 \ \text{ for all } \ 1 \leq i,j \leq n$$

Then

$$\begin{aligned}
&\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \leq u_j\}\Big) \\
\leq \ &\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \leq u_j\}\Big) \leq \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \leq u_j\}\Big) \\
&\exp\Big\{\sum_{1\leq i<j\leq n} \ln\Big(\frac{\pi - 2\arcsin(r_{ij}^0)}{\pi - 2\arcsin(r_{ij}^1)}\Big)\exp\Big(-\frac{(u_i^2+u_j^2)}{2(1+r_{ij}^1)}\Big)\Big\}
\end{aligned}$$

for any $u_i \geq 0, i = 1, 2, \cdots, n$ satisfying

$$(r_{ki}^l - r_{ij}^l r_{kj}^l)u_i + (r_{kj}^l - r_{ij}^l r_{ki}^l)u_j \geq 0$$

for $l = 0, 1$ and for all $1 \leq i, j, k \leq n$.

In particular, for $u \geq 0$

$$\begin{aligned}
&\mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \leq u\}\Big) \\
\leq\ & \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\xi_j \leq u\}\Big) \leq \mathbb{P}\Big(\bigcap_{j=1}^{n}\{\eta_j \leq u\}\Big) \\
& \exp\Big\{ \sum_{1\leq i<j\leq n} \ln\Big(\frac{\pi - 2\arcsin(r_{ij}^0)}{\pi - 2\arcsin(r_{ij}^1)}\Big) \exp\Big(-\frac{u^2}{1+r_{ij}^1}\Big)\Big\}
\end{aligned}$$

5. Wenbo's weak correlation inequality

Let X and Y be centered jointly Gaussian vectors in a separable Banach space E.

- Gaussian correlation conjecture: For any symmetric convex sets A and B in E

$$P(X \in A,\ Y \in B) \geq P(X \in A)P(Y \in B)$$

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- Li (1999): For any $0 < \lambda < 1$

$$P(X \in A,\ Y \in B) \geq P(X \in \lambda A)P(Y \in (1-\lambda^2)^{1/2}B)$$


Thank You